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Application of Helmholtz/Hodge Decomposition to Finite Element Methods for Two-Dimensional Maxwell's Equations

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APPLICATION OF HELMHOLTZ/HODGE DECOMPOSITION TO FINITE ELEMENT METHODS FOR TWO-DIMENSIONAL MAXWELL'S EQUATIONS

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Abstract

In this work we apply the two-dimensional Helmholtz/Hodge decomposition to develop new finite element schemes for two-dimensional Maxwell's equations. We begin with the introduction of Maxwell's equations and a brief survey of finite element methods for Maxwell's equations. Then we review the related fundamentals in Chapter 2. In Chapter 3, we discuss the related vector function spaces and the Helmholtz/Hodge decomposition which are used in Chapter 4 and 5. The new results in this dissertation are presented in Chapter 4 and Chapter 5. In Chapter 4, we propose a new numerical approach for two-dimensional Maxwell's equations that is based on the Helmholtz/Hodge decomposition for divergence-free vector fields. In this approach an approximate solution for Maxwell's equations can be obtained by solving standard second order scalar elliptic boundary value problems. This new approach is illustrated by a P_1 finite element method. In Chapter 5, we further extend the new approach described in Chapter 4 to the interface problem for Maxwell's equations. We use the extraction formulas and multigrid method to overcome the low regularity of the solution for the Maxwell interface problem. The theoretical results obtained in this dissertation are confirmed by numerical experiments.

Chapter 1
Introduction

1.1 Maxwell's Equations and the Corresponding Interface Problems

In this section we introduce several formulations of Maxwell equations, their boundary conditions and interface conditions. It is mainly based on the books [60, 7, 49, 40].

1.1.1 Maxwell's Equations in Integral Form

Consider an open surface S bounded by a closed contour C. The first two Maxwell's equations are given in the following equations

$$\oint_C \boldsymbol{E}(\boldsymbol{x}, t) \cdot dl = -\frac{d}{dt} \iint_S \boldsymbol{B}(\boldsymbol{x}, t) \cdot dS \tag{1.1.1}$$

and

$$\oint_C \boldsymbol{H}(\boldsymbol{x}, t) \cdot dl = \frac{d}{dt} \iint_S \boldsymbol{D}(\boldsymbol{x}, t) \cdot dS + \iint_S \boldsymbol{J} \cdot dS, \tag{1.1.2}$$

where

$$\boldsymbol{E} = \text{electric field intensity},$$

$$\boldsymbol{D} = \text{electric displacement},$$

$$\boldsymbol{B} = \text{magnetic induction},$$

$$\boldsymbol{H} = \text{magnetic field intensity},$$

and

$$\boldsymbol{J} = \text{electric current density}.$$

Next, consider a volume V enclosed by a surface S. Two other Maxwell's equations are given by the following equations

$$\iint_S \boldsymbol{D}(\boldsymbol{x}, t) \cdot \mathrm{d}S = \iiint_V \rho(\boldsymbol{x}, t) \mathrm{d}V \tag{1.1.3}$$

and

$$\iint_S \boldsymbol{B}(\boldsymbol{x}, t) \cdot \mathrm{d}S = 0, \tag{1.1.4}$$

where

$$\rho = \text{electric charge density in } V.$$

Remark 1.1.1. The integral form of Maxwell's equations (1.1.1)-(1.1.4) is valid everywhere. We will use them to derive interface conditions.

1.1.2 Maxwell's Equations in Differential Form

By applying Stokes' theorem and Gauss' theorem from calculus (cf. [57]), we can convert Maxwell's equations in integral form into Maxwell's equations in differential form.

By applying Stokes' theorem to (1.1.1) and (1.1.2), we get

$$\iint_S \boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{x}, t) \cdot \mathrm{d}S = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \boldsymbol{B} \cdot \mathrm{d}S \tag{1.1.5}$$

and

$$\iint_S \boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{x}, t) \cdot \mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}t} \iint_S \boldsymbol{E}(\boldsymbol{x}, t) \cdot dS + \iint_S \boldsymbol{J} \cdot dS. \tag{1.1.6}$$

Because of the arbitrariness of the surface S, equations (1.1.5) and (1.1.6) lead to the following differential equations

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{x}, t) = -\frac{\partial}{\partial t} \boldsymbol{B}(\boldsymbol{x}, t) \tag{1.1.7}$$

and

$$\nabla \times \boldsymbol{H}(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\boldsymbol{D}(\boldsymbol{x}, t) + \boldsymbol{J}. \tag{1.1.8}$$

By applying Gauss' theorem to equations (1.1.3) and (1.1.4), we obtain

$$\nabla \cdot \boldsymbol{D} = \rho \tag{1.1.9}$$

and

$$\nabla \cdot \boldsymbol{B} = 0. \tag{1.1.10}$$

By taking the divergence of (1.1.8), applying (1.1.9) and the vector identity (cf. [57])

$$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0 \quad \text{for a smooth vector field } \boldsymbol{v},$$

we obtain the conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0. \tag{1.1.11}$$

1.1.3 Constitutive Relations

A medium affects electromagnetic fields through three phenomena: electric polarization, magnetic polarization, and electric conduction. Electric polarization leads to the constitutive relation for the electric field. In most cases it can be expressed as

$$\boldsymbol{D} = \epsilon \boldsymbol{E}, \tag{1.1.12}$$

where D is called the electric flux density and ϵ is called the permittivity of the dielectric medium. Magnetic polarization leads to the constitutive relation for the magnetic field. In most materials it can be expressed as

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{1.1.13}$$

where $\boldsymbol{H}$ is called the magnetic field intensity and μ is called the permeability of the material. The electric conduction leads to the final constitutive relation

$$\boldsymbol{J}_c = \sigma \boldsymbol{E}, \tag{1.1.14}$$

where σ is called the conductivity and $\boldsymbol{J}_c$ is called the conduction current, which can be regarded as a part of the total electric current.

1.1.4 Boundary Conditions and Interface Conditions

It is sufficient to consider interface conditions since the boundary of the domain is a special type of interface. Without loss of generality, we only consider an interface between two different mediums. Moreover, we assume a surface charge distribution over the interface. The surface charge density is defined as the amount of charge over a unit area on the surface. Applying (1.1.3) in a small cylinder with one of its faces in medium 1 and the other in medium 2 and letting its thickness $\Delta t \to 0$, we obtain

$$\boldsymbol{D}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{D}_2 \cdot \boldsymbol{n}_2 = \rho_{e,s}, \tag{1.1.15}$$

where $\rho_{e,s}$ denotes the surface electric charge density, $\boldsymbol{n}_i \quad (i = 1, 2)$ denotes the normal direction of the boundary of medium i, $\boldsymbol{D}_i \quad (i = 1, 2)$ denotes the electric displacement in medium i.

Using a similar strategy, we obtain

$$\boldsymbol{B}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{B}_2 \cdot \boldsymbol{n}_2 = 0, \tag{1.1.16}$$

$$\boldsymbol{H}_1 \times \boldsymbol{n}_1 + \boldsymbol{H}_2 \times \boldsymbol{n}_2 = \boldsymbol{J}_s, \tag{1.1.17}$$

and

$$\boldsymbol{E}_1 \times \boldsymbol{n}_1 + \boldsymbol{E}_2 \times \boldsymbol{n}_2 = \boldsymbol{0}, \tag{1.1.18}$$

where $\boldsymbol{J}_s$ denotes the surface current density, $\boldsymbol{E}_i$ and $\boldsymbol{H}_i \quad (i = 1, 2)$ denote the electric and magnetic field intensity in medium i. Here we define a notation to simplify the description of the interface conditions.

Notation 1.1.2. *For a vector field* $\boldsymbol{u}$ *and* Γ *the interface between two mediums, let us denote*

$$[\boldsymbol{u} \cdot \boldsymbol{n}]|_\Gamma := \boldsymbol{u}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{u}_2 \cdot \boldsymbol{n}_2 \quad \textit{or} \quad [\boldsymbol{u} \cdot \boldsymbol{n}] := \boldsymbol{u}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{u}_2 \cdot \boldsymbol{n}_2 \quad \textit{on the interface } \Gamma$$

and

$$[\boldsymbol{u} \times \boldsymbol{n}]|_\Gamma := \boldsymbol{u}_1 \times \boldsymbol{n}_1 + \boldsymbol{u}_2 \times \boldsymbol{n}_2 \quad \textit{or} \quad [\boldsymbol{u} \times \boldsymbol{n}] := \boldsymbol{u}_1 \times \boldsymbol{n}_1 + \boldsymbol{u}_2 \times \boldsymbol{n}_2 \quad \textit{on the interface } \Gamma,$$

where $\boldsymbol{n}_i$ *denotes the normal direction of the interface with respect to medium* i.

Using this new notation, the above interface conditions can be written as

$$[\boldsymbol{D} \cdot \boldsymbol{n}]|_\Gamma = \rho_{e,s}, \tag{1.1.19}$$

$$[\boldsymbol{B} \cdot \boldsymbol{n}]|_\Gamma = 0, \tag{1.1.20}$$

$$[\boldsymbol{E} \times \boldsymbol{n}]|_\Gamma = \boldsymbol{0}, \tag{1.1.21}$$

$$[\boldsymbol{H} \times \boldsymbol{n}]|_\Gamma = \boldsymbol{J}_s. \tag{1.1.22}$$

1.1.5 Time Harmonic Maxwell's Equations

Now we derive time-harmonic Maxwell's equations from the differential formulation of Maxwell's equations (1.1.7)-(1.1.10).

Assume that functions and vector fields in Maxwell's equations have the form

$$\boldsymbol{E} = \Re(\hat{\boldsymbol{E}}(\boldsymbol{x})\mathsf{exp}(-i\omega t)), \tag{1.1.23}$$

$$\boldsymbol{D} = \Re(\hat{\boldsymbol{D}}(\boldsymbol{x})\mathsf{exp}(-i\omega t)), \tag{1.1.24}$$

$$\boldsymbol{H} = \Re(\hat{\boldsymbol{H}}(\boldsymbol{x})\mathsf{exp}(-i\omega t)), \tag{1.1.25}$$

$$\boldsymbol{B} = \Re(\hat{\boldsymbol{B}}(\boldsymbol{x})\mathsf{exp}(-i\omega t)), \tag{1.1.26}$$

$$\boldsymbol{J} = \Re(\hat{\boldsymbol{J}}(\boldsymbol{x})\mathsf{exp}(-i\omega t)), \tag{1.1.27}$$

$$\rho = \Re(\hat{\rho}(\boldsymbol{x})\mathsf{exp}(-i\omega t)), \tag{1.1.28}$$

where $i = \sqrt{-1}$ and $\Re(\cdot)$ denotes the real part of the expression in parentheses. After substituting the relations (1.1.23)- (1.1.28) into (1.1.7)-(1.1.10), we get the time-harmonic Maxwell system:

$$\nabla \times \hat{\boldsymbol{H}} = -i\omega\hat{\boldsymbol{D}} + \hat{\boldsymbol{J}}, \tag{1.1.29}$$

$$\nabla \times \hat{\boldsymbol{E}} = i\omega\hat{\boldsymbol{B}}, \tag{1.1.30}$$

$$\nabla \cdot \hat{\boldsymbol{D}} = \hat{\rho}, \tag{1.1.31}$$

$$\nabla \cdot \hat{\boldsymbol{B}} = 0. \tag{1.1.32}$$

Combining the constitutive relations (1.1.12)-(1.1.13), we can eliminate $\hat{\boldsymbol{D}}$, $\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{H}}$ from (1.1.29)-(1.1.32) and obtain the following equation:

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{u}) - \omega^2\epsilon\boldsymbol{u} = \boldsymbol{f}, \tag{1.1.33}$$

where $\boldsymbol{u} = \hat{\boldsymbol{E}}$ and $\boldsymbol{f} = i\omega\hat{\boldsymbol{J}}$.

Correspondingly, the interface conditions (1.1.19)- (1.1.22) imply the following interface conditions for the solution $\boldsymbol{u}$ of (1.1.33):

$$[(\epsilon\hat{\boldsymbol{u}}) \cdot \boldsymbol{n}]|_\Gamma = \hat{\rho}_{e,s}, \tag{1.1.34}$$

$$[(\nabla \times \hat{\boldsymbol{u}}) \cdot \boldsymbol{n}]|_\Gamma = 0, \tag{1.1.35}$$

$$[\hat{\boldsymbol{u}} \times \boldsymbol{n}]|_\Gamma = \boldsymbol{0}, \tag{1.1.36}$$

$$[(\mu^{-1}\nabla \times \hat{\boldsymbol{u}}) \times \boldsymbol{n}]|_\Gamma = \hat{\boldsymbol{J}}_s, \tag{1.1.37}$$

where $\rho_{e,s} = \Re(\hat{\rho}_{e,s}\mathsf{exp}(-i\omega t))$ and $\boldsymbol{J}_s = \Re(\hat{\boldsymbol{J}}_s\mathsf{exp}(-i\omega t))$.

1.1.6 Two-dimensional Maxwell's Equations

In many cases, Maxwell's equations can be reduced to a two dimensional problem. For example, in the case the region where an electromagnetic field exists is a cylindrical body, the cross section of the cylinder is orthogonal to z-axis, and the electric field is orthogonal to z-axis and independent of the z variable, we could write electric field and magnetic field as

$$\boldsymbol{E} = (E_x(x,y), E_y(x,y), 0),$$

and correspondingly,

$$\boldsymbol{H} = (0, 0, H_z(x,y)).$$

In this situation, we will get a two-dimensional version of the equation (1.1.33):

$$\nabla \times (\mu^{-1} \nabla \times \boldsymbol{u}) - \omega^2 \epsilon \boldsymbol{u} = \boldsymbol{f}, \tag{1.1.38}$$

where $\boldsymbol{u}$ is a two-dimensional vector field.

1.1.7 Weak Formulation of Maxwell's Interface Problems

In this section we derive the weak formulation for certain Maxwell's interface problems. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\Omega_j, 1 \le j \le J$ be polygonal subdomains of Ω that form a partition of Ω (See Subsection 2.2.2 for more details). Γ denotes the interface between Ω_j's. Suppose that the vector function $\boldsymbol{u}$ satisfying

$$\boldsymbol{u}|_{\Omega_j} \in [\mathcal{C}^2(\Omega_j)]^2 \cap [\mathcal{C}^1(\bar{\Omega}_j)]^2 \quad 1 \le j \le J$$

is the classical solution of Maxwell's interface problem:

$$\nabla \times (\mu_j^{-1} \nabla \times \boldsymbol{u}_j) - k^2 \epsilon_j \boldsymbol{u}_j = \boldsymbol{f}_j \quad \text{in the domain } \Omega_j, 1 \le j \le J, \tag{1.1.39a}$$

$$\boldsymbol{n} \times \boldsymbol{u} = 0 \quad \text{on the boundary } \partial\Omega, \tag{1.1.39b}$$

$$[\boldsymbol{n} \times \boldsymbol{u}] = 0 \quad \text{on the interface } \Gamma, \tag{1.1.39c}$$

$$\nabla \cdot (\epsilon \boldsymbol{u}) = 0 \quad \text{in } \Omega_j, 1 \le j \le J, \tag{1.1.39d}$$

$$[\boldsymbol{n} \cdot \epsilon \boldsymbol{u}] = 0 \quad \text{on the interface } \Gamma, \tag{1.1.39e}$$

$$[\mu^{-1}\boldsymbol{\nabla} \times \boldsymbol{u}] = 0 \quad \text{on the interface } \Gamma, \tag{1.1.39f}$$

where $\boldsymbol{f}_j$ is smooth in the closed subdomain $\bar{\Omega}_j, 1 \leq j \leq J$. We attempt to find a proper weak formulation of the above interface problem (1.1.39). Let $\boldsymbol{v}$ be an arbitrary vector function in $\mathbb{R}^2$ satisfying

$$\boldsymbol{v}|_{\Omega_j} \in \left[\mathcal{C}^1(\Omega_j)\right]^2 \cap \left[\mathcal{C}^0(\bar{\Omega}_j)\right]^2 \quad 1 \leq j \leq J.$$

Taking the dot product on both sides of (1.1.39a) by the vector function $\boldsymbol{v}|_{\Omega_j}$, integrating over the subdomain Ω_j, we have

$$\int_{\Omega_j} \boldsymbol{\nabla} \times (\mu_j^{-1}\boldsymbol{\nabla} \times \boldsymbol{u}_j) \cdot \boldsymbol{v} dx - \int_{\Omega_j} k^2 \epsilon_j \boldsymbol{u}_j \cdot \boldsymbol{v} dx = \int_{\Omega_j} \boldsymbol{f}_j \cdot \boldsymbol{v} dx. \tag{1.1.40}$$

Hence,

$$\sum_{j=1}^{J} (\int_{\Omega_j} \boldsymbol{\nabla} \times (\mu_j^{-1}\boldsymbol{\nabla} \times \boldsymbol{u}_j) \cdot \boldsymbol{v} dx - \int_{\Omega_j} k^2 \epsilon_j \boldsymbol{u}_j \cdot \boldsymbol{v} dx) = \sum_{j=1}^{J} \int_{\Omega_j} \boldsymbol{f}_j \cdot \boldsymbol{v} dx. \tag{1.1.41}$$

Using integration by parts, we have that

$$\int_{\Omega_j} \boldsymbol{\nabla}\times(\mu_j^{-1}\boldsymbol{\nabla}\times\boldsymbol{u}_j)\cdot\boldsymbol{v} dx = \int_{\Omega_j} \mu_j^{-1}(\boldsymbol{\nabla}\times\boldsymbol{u})\cdot(\boldsymbol{\nabla}\times\boldsymbol{v}) dx + \int_{\Omega_j} \mu_j^{-1}(\boldsymbol{\nabla}\times\boldsymbol{v}_j)\cdot(\boldsymbol{v}\times\boldsymbol{n}) ds. \tag{1.1.42}$$

Therefore, (1.1.41) and (1.1.42) lead to

$$\sum_{j=1}^{J} \int_{\Omega_j} (\mu_j^{-1}\boldsymbol{\nabla}\times\boldsymbol{u}_j\cdot\boldsymbol{v} - k^2\epsilon_j\boldsymbol{u}_j\cdot\boldsymbol{v}) dx + \sum_{j=1}^{J} \int_{\partial\Omega_j} \mu_j^{-1}(\boldsymbol{\nabla}\times\boldsymbol{u}_j)\cdot(\boldsymbol{v}\times\boldsymbol{n}) ds = \sum_{j=1}^{J} \int_{\Omega_j} \boldsymbol{f}_j\cdot\boldsymbol{v} dx, \tag{1.1.43}$$

or equivalently,

$$\begin{aligned} \sum_{j=1}^{J} \int_{\Omega_j} (\mu_j^{-1}\boldsymbol{\nabla} \times \boldsymbol{u}_j \cdot \boldsymbol{v} - k^2\epsilon_j\boldsymbol{u}_j \cdot \boldsymbol{v}) dx + \sum_{j=1}^{J} (\int_{\partial\Omega} \mu^{-1}(\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot (\boldsymbol{v} \times \boldsymbol{n}) ds \\ + \int_{\Gamma} \mu^{-1}(\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot [\boldsymbol{v} \times \boldsymbol{n}]\, ds) = \sum_{j=1}^{J} \int_{\Omega_j} \boldsymbol{f}_j \cdot \boldsymbol{v} dx. \end{aligned} \tag{1.1.44}$$

By the homogeneous boundary and interface conditions (1.1.39b), (1.1.39c) and (1.1.39f), (1.1.44) implies that

$$\sum_{j=1}^{J}\int_{\Omega_j}(\mu_j^{-1}\boldsymbol{\nabla}\times\boldsymbol{u}_j\cdot\boldsymbol{v}-k^2\epsilon_j\boldsymbol{u}_j\cdot\boldsymbol{v})dx=\sum_{j=1}^{J}\int_{\Omega_j}\boldsymbol{f}_j\cdot\boldsymbol{v}dx, \tag{1.1.45}$$

or,

$$(\mu^{-1}\boldsymbol{\nabla}\times\boldsymbol{u},\boldsymbol{\nabla}\times\boldsymbol{v})-k^2(\epsilon\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}), \tag{1.1.46}$$

where $(\cdot,\cdot)$ denotes the summation of the L_2 inner product on $\Omega_j, 1\le j\le J$. A natural choice of the variational space for the weak formulation (1.1.46) will be $H_0(curl;\Omega)\cap H(div^0;\Omega;\epsilon)$.

So we are considering the weak formulation of Maxwell's interface problem: Find $\boldsymbol{u}\in H_0(curl;\Omega)\cap H(div^0;\Omega;\epsilon)$ such that

$$(\mu^{-1}\boldsymbol{\nabla}\times\boldsymbol{u},\boldsymbol{\nabla}\times\boldsymbol{v})-k^2(\epsilon\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v})\quad\forall\boldsymbol{v}\in H_0(curl;\Omega)\cap H(div^0;\Omega;\epsilon). \tag{1.1.47}$$

Remark 1.1.3. The formal definition of the space $H_0(curl;\Omega)\cap H(div^0;\Omega;\epsilon)$ will be introduced in Chapter 3. For the moment, we just use it.

Remark 1.1.4. Here we consider Maxwell interface problem with homogeneous interface conditions, which means we assume there are no interface charge and interface current. The case where that there are interface charge and interface current can be reduced to the problem we consider here.

1.2 History of Finite Element Methods for Maxwell's Equations and the Corresponding Interface Problems

The natural choice of variational space for the variational problem of Maxwell's equations is $H_0(curl;\Omega)\cap H(div^0;\Omega;\epsilon)$. However, every conforming finite element subspace in $H_0(curl;\Omega)\cap H(div^0;\Omega;\epsilon)$ must be in $[H^1(\Omega)]^d$ ($d=2$ or 3), since it consists of continuous piecewise polynomials, and the intersection of $[H^1(\Omega)]^d$ and

$[H_0(curl;\Omega) \cap H(div^0;\Omega;\epsilon)]$ is a proper closed subspace of $H_0(curl;\Omega) \cap H(div^0;\Omega;\epsilon)$ when the domain Ω has re-entrant corners [49]. Therefore, the resulting finite element space is not dense in $H_0(curl;\Omega) \cap H(div^0;\Omega;\epsilon)$ as the mesh size goes to zero and hence the finite element solution may not converge to the exact $H_0(curl;\Omega) \cap H(div^0;\Omega;\epsilon)$ solution [49]. Instead, some people use the larger space $H_0(curl;\Omega)$ as the variational space and solve the curl-curl variational problem for Maxwell's equations by $H(curl)$-conforming edge elements [39, 49, 51, 52]. More recently, successful algorithms have been discovered for this curl-curl problem that either solve a curl-curl and grad-div problem using nodal H^1 vector finite elements complemented by singular vector fields [31], or solve its regularized version using standard nodal H^1 vector finite elements [27]. Alternatively one can use nonconforming methods [15, 17, 18, 13, 19]. However, in the works we mentioned above, the dielectric and magnetic permeability were assumed to be constant. In this dissertation we will consider the case where the dielectric and magnetic permeability are piecewise constant. The main challenge in this situation is that the regularity of the solution could be much worse [28], and hence most of the existing methods fail. In order to develop a successful algorithm for Maxwell's equations in heterogeneous media, a new algorithm for the homogeneous media case was first proposed in our work [14], which is based on Hodge decomposition. Following this new approach, an adaptive P_1 finite element method have been carried out in [16]. In this dissertation, we further extend this new approach to the heterogeneous media by exploiting extraction formulas and full multigrid methods [12, 21, 22, 23].

Chapter 2
Fundamentals

2.1 Sobolev Spaces

In this section we review some basic facts about Sobolev spaces. They are based on the references [1, 33, 34, 36]. First, let us define the notations for derivatives and related function spaces. Assume $u : \Omega \to \mathbb{R}, \quad x \in \Omega$, where Ω is a bounded open set in $\mathbb{R}^d, \quad d = 2$ or 3.

Notation 2.1.1. *A vector of the form $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where each component is a nonnegative integer, is called a multi-index of order $|\alpha| = \sum_{i=1}^{n} \alpha_i$. Then we denote*

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Notation 2.1.2. *The following function spaces are denoted by $\mathcal{D}(\Omega)$, $\mathcal{D}(\bar{\Omega})$, $\mathcal{C}^0(\bar{\Omega})$, respectively:*

$$\mathcal{D}(\Omega) := \{v : v \textit{ is smooth with compact support in the domain } \Omega\},$$

$$\begin{aligned} \mathcal{D}(\bar{\Omega}) :=& \{v|_{\bar{\Omega}} : v|_{\bar{\Omega}} \textit{ is the restriction to } \Omega \textit{ of a smooth function } v \\ & \textit{with compact support in } \mathbb{R}^d\}, \end{aligned}$$

$$\mathcal{C}^0(\bar{\Omega}) := \{v : v \textit{ is continuous in the domain } \bar{\Omega}\}.$$

We also need the concept of weak derivatives.

Definition 2.1.3. Suppose u is locally integrable in Ω, and α is a multi-index. If there exists a locally integrable function v such that

$$\int_\Omega \phi v dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx \quad \forall \phi \in \mathcal{D}(\Omega),$$

then v is called the α^{th} weak derivative of u, written as $D^\alpha u = v$.

Now let us define the Sobolev spaces.

Definition 2.1.4. Let k be a nonnegative integer. The Sobolev space $H^k(\Omega)$ is defined as follows:

$$H^k(\Omega) = \{u \in L_2(\Omega) : D^\alpha u \in L_2(\Omega), \text{ for all } |\alpha| \le k\}.$$

We define the subspace $H_0^k(\Omega)$ of $H^k(\Omega)$ by

$$H_0^k(\Omega) = \text{ the closure of } \mathcal{D}(\Omega) \text{ in } H^k(\Omega).$$

Definition 2.1.5. Let $s = k+\sigma$, where k is a nonnegative integer, and $0 < \sigma < 1$. The fractional order Sobolev space $H^s(\Omega)$ is defined as follows:

$$H^s(\Omega) = \{u \in L_2(\Omega) : u \in H^k(\Omega) \text{ and } \iint_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{d+2s}} dxdy < \infty, \quad \forall |\alpha| = k\}.$$

Next, we will discuss some properties of the Sobolev spaces $H^k(\Omega)$.

Theorem 2.1.6. *For any nonnegative integer k, $H^k(\Omega)$ is a Hilbert space with the inner product*

$$(u,v)_{H^k} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha u \cdot D^\alpha v dx \quad \forall u, v \in H^k(\Omega)$$

and the induced norm

$$\|u\|_{H^k(\Omega)} = \{\sum_{|\alpha|\le k} \|D^\alpha u\|^2_{L_2(\Omega)}\}^{1/2}.$$

For any positive number $s = k+\sigma$, where k is a nonnegative integer and $0 < \sigma < 1$, $H^s(\Omega)$ is a Hilbert space with the inner product

$$(u,v)_{H^s(\Omega)} = (u,v)_{H^k(\Omega)} + \sum_{|\alpha|=k} \iint_{\Omega\times\Omega} \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha v(x) - D^\alpha v(y))}{|x-y|^{d+2s}} dxdy.$$

To better understand $H^k(\Omega)$, we need a density property of $H^k(\Omega)$. It turns out that the density property and many other properties of Sobolev spaces depend on the regularity of the domain Ω. So let us first define a geometrical condition on the domain which will be sufficient for our subsequent purposes whenever the regularity of the boundary of the domain is needed.

Definition 2.1.7. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with the boundary Γ. Then we say that the domain Ω has a *Lipschitz continuous* boundary if, for any point $x^0 \in \Gamma$, there exist $r > 0$ and a Lipschitz continuous function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$, up to relabeling and reorienting the coordinates axes if necessary, such that

$$\Omega \cap B(x^0, r) = \{x \in B(x^0, r) : x_d > \gamma(x_1, x_2, \cdots, x_{d-1})\},$$

where $B(x^0, r) = \{x \in \mathbb{R}^d : \|x - x^0\| < r\}$ and $x = (x_1, x_2, \cdots, x_{d-1}, x_d)$. Similarly, we say that the domain Ω has a $\mathcal{C}^1$ *continuous* boundary if, for any point $x^0 \in \Gamma$, there exist $r > 0$ and a $\mathcal{C}^1$ continuous function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega \cap B(x^0, r) = \{x \in B(x^0, r) : x_d > \gamma(x_1, x_2, \cdots, x_{d-1})\}.$$

Theorem 2.1.8. (Approximation by Smooth Functions on $\mathbb{R}^d$) *Suppose that Ω has a Lipschitz continuous boundary, then $\mathcal{D}(\bar{\Omega})$ is dense in $H^k(\Omega)$.*

To keep things simple, we will only state special cases of the Sobolev embedding theorems which will be needed in later chapters.

Theorem 2.1.9. (Embedding Theorem) *Suppose that the domain $\Omega \subset \mathbb{R}^2$ has a Lipschitz continuous boundary. Then $H^1(\Omega)$ is compactly embedded in $L_q(\Omega)$ for $q \geq 1$, and $H^2(\Omega)$ is embedded in $\mathcal{C}^{0,\eta}(\bar{\Omega})$ for $0 < \eta < 1$, where $\mathcal{C}^{0,\eta}(\bar{\Omega})$ are the Hölder spaces defined by*

$$\mathcal{C}^{0,\eta}(\bar{\Omega}) = \{u \in C^0(\bar{\Omega}) : \sup_{\substack{x \neq y, \\ x, y \in \bar{\Omega}}} \frac{|u(x) - u(y)|}{|x - y|^\eta} < \infty\} \tag{2.1.1}$$

and their corresponding norms are defined by

$$\|u\|_{C^{0,\eta}(\bar{\Omega})} = \max_{x\in\bar{\Omega}}|u(x)| + \sup_{\substack{x\neq y\\ x,y\in\bar{\Omega}}} \frac{|u(x)-u(y)|}{|x-y|^{\eta}}. \tag{2.1.2}$$

Theorem 2.1.10. (Trace Theorem) *Suppose that $\Omega \subset \mathbb{R}^2$ has a $\mathcal{C}^1$ continuous boundary Γ. Then there exists a linear operator*

$$T : H^1(\Omega) \to L^2(\Gamma)$$

such that

$$Tu = u|_{\Gamma} \quad \forall u \in \mathcal{D}(\bar{\Omega}).$$

Moreover,

$$\|Tu\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega),$$

where the constant C depends on Ω.

Remark 2.1.11. The trace theorem can be extended to polygonal domains. For details, see the reference [36].

2.2 Regularity Results

2.2.1 Regularity of Elliptic Problems

In this subsection, we consider the regularity of elliptic problems with homogeneous Dirichlet boundary condition and Neumann boundary condition in nonconvex domains. The main results below are from the references [44, 36, 29, 50].

Suppose that Ω is a polygonal domain in $\mathbb{R}^2$. Let $c_1, c_2, \cdots, c_{N_\Omega}$ be the corners of Ω, $\omega_1, \omega_2, \cdots, \omega_{N_\Omega}$ be the interior angles around those corners, and $\omega = \max\{\omega_1, \omega_2, \cdots, \omega_{N_\Omega}\}$. For $f \in L_2(\Omega)$, consider the Dirichlet problem:

Find $u \in H_0^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in the domain } \Omega, \tag{2.2.1a}$$

$$u = 0 \quad \text{on the boundary } \partial\Omega. \tag{2.2.1b}$$

The regularity of the problem (2.2.1) is stated in the following theorem (cf. [29, 36]).

Theorem 2.2.1. (Regularity of the Dirichlet Problem of the Poisson Equation) *Suppose that Ω is nonconvex, i.e., $\omega > \pi$. Then the solution u of (2.2.1) can be decomposed into a singular part u_S and a regular part u_R, or equivalently, $u = u_S + u_R$, where $u_R \in H^2(\Omega)$. Moreover, there exist constants κ_l, for $\omega_l > \pi$, such that*

$$u_S = \sum_{\omega_l > \pi} \kappa_l s_l, \tag{2.2.2}$$

and

$$s_l = r^{\frac{\pi}{\omega_l}} \sin(\frac{\pi}{\omega_l}\theta)\varrho_l(r)$$

is a function defined with respect to the polar coordinates (r, θ) around the corner c_l and $\varrho_l(r)$ is a cut-off function which equals 1 near the corner and 0 away from the corner. We also have the elliptic regularity estimate

$$\|u_R\|_{H^2(\Omega)} + \sum_{\omega_l > \pi} |\kappa_l| \leq C\|f\|_{L_2(\Omega)}. \tag{2.2.3}$$

Under the same assumption for the domain Ω and f as in the Dirichlet problem, we consider the Neumann problem:
Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in the domain } \Omega, \tag{2.2.4a}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on the boundary} \partial\Omega. \tag{2.2.4b}$$

The regularity of the problem (2.2.4) is stated in the following theorem (cf. [29, 36]).

Theorem 2.2.2. (Regularity of the Neumann Problem of the Poisson Equation) *Suppose that Ω is nonconvex, i.e., $\omega > \pi$. Then the solution u of (2.2.1) can be* *decomposed into a singular part u_S and a regular part u_R, or equivalently, $u = u_S + u_R$, where $u_R \in H^2(\Omega)$. Moreover, there exist constants κ_l, for $w_l > \pi$, such that*

$$u_S = \sum_{\omega_l > \pi} \kappa_l s_l, \tag{2.2.5}$$

where

$$s_l = r^{\frac{\pi}{\omega_l}} \cos(\frac{\pi}{\omega_l}\theta)\varrho_l(r)$$

is a function defined with respect to the polar coordinates (r, θ) around the corner c_l and $\varrho_l(r)$ is a cut-off function which equals 1 near the corner and 0 away from the corner. We also have the elliptic regularity estimate

$$\|u_R\|_{H^2(\Omega)} + \sum_{\omega_l > \pi} |\kappa_l| \le C\|f\|_{L_2(\Omega)}. \tag{2.2.6}$$

Remark 2.2.3. The functions s_l in (2.2.2) and (2.2.5) are called singular functions. The constants κ_l in (2.2.2) and (2.2.5) are called stress intensity factors.

2.2.2 Regularity of Elliptic Interface Problems

In this subsection we discuss the regularity of elliptic interface problems. The main references are [41, 42, 43, 46, 6, 32, 53, 54, 55, 56].

Suppose that Ω is a polygonal domain in $\mathbb{R}^2$, and $\Omega_j, 1 \le j \le J$, are polygonal subdomains of Ω that form a partition of Ω (See Figure 2.1), i.e.,

$$\Omega_{j_1} \cap \Omega_{j_2} = \emptyset \quad \text{for } j_1 \ne j_2 \text{ and } \cup_{j=1}^{J} \bar{\Omega}_j = \bar{\Omega}.$$

Let $f \in L_2(\Omega)$, ρ_j be positive constants and $\rho : \Omega \to \mathbb{R}$ be a function defined by

$$\rho(x) = \rho_j \quad \forall x \in \Omega_j, 1 \le j \le J.$$

Denote the interface between the subdomains Ω_j by Γ.

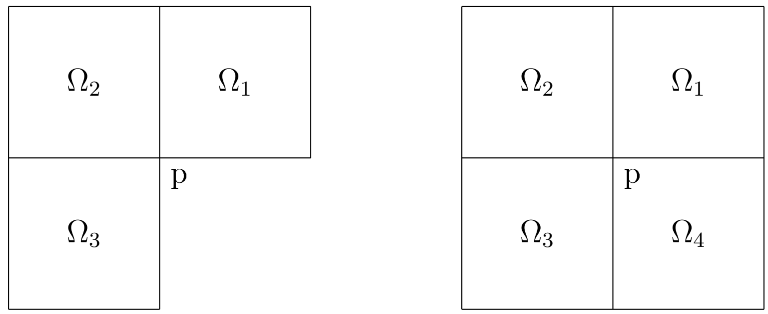

FIGURE 2.1. Examples of the domain Ω.

Consider the following elliptic interface problem with Neumann boundary conditions:

Find u such that

$$-\rho_j \Delta u = f \quad \text{in} \quad \Omega_j, 1 \le j \le J, \tag{2.2.7a}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on the boundary} \quad \partial\Omega, \tag{2.2.7b}$$

$$[u] = 0 \quad \text{on the interface} \quad \Gamma, \tag{2.2.7c}$$

$$\left[\rho \frac{\partial u}{\partial n}\right] = 0 \quad \text{on the interface} \quad \Gamma. \tag{2.2.7d}$$

Here $[u]$ denotes the jump of u and $[u] = 0$ on the interface Γ means that u is continuous across Γ and $\left[\rho \frac{\partial u}{\partial n}\right]$ denotes

$$\left[\rho \frac{\partial u}{\partial n}\right] = \rho_- \frac{\partial u}{\partial n_-} + \rho_+ \frac{\partial u}{\partial n_+},$$

where ρ_- (resp. ρ_+) denotes the weight ρ in the subdomain Ω_- (resp. Ω_+) and n_- (resp. n_+) denotes the unit normal along the interface Γ when we view Γ as the boundary of the subdomain Ω_- (resp. Ω_+).

Define the weak bilinear form $a_\rho(\cdot, \cdot)$ by

$$a_\rho(u, v) = \int_\Omega \rho \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v dx \quad \forall u, v \in H^1(\Omega).$$

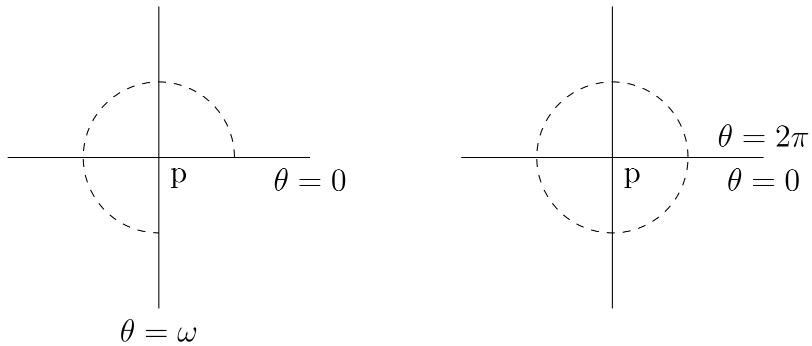

FIGURE 2.2. Polar coordinates for the Sturm-Liouville problems.

Then the weak form of the interface problem is:

Find $u \in H^1(\Omega)$ such that

$$a_\rho(u, v) = (f, v) \quad \forall v \in H^1(\Omega). \tag{2.2.8}$$

Away from the vertices of $\Omega_1, \cdots, \Omega_J$, the solution u of (2.2.8) has the standard regularity. In other words, $u \in H^2(\Omega_{j,\delta})$ for $1 \le j \le J$, where $\Omega_{j,\delta}$ is obtained from Ω_j by excising the closure of a disc $D(p, \delta)$ ($\delta > 0$ is arbitrary) around the vertex p. At a vertex p common to more than one subdomain (i.e., an interface vertex), the solution u of (2.2.8) is in general singular in the sense that it does not belong to $H^2(D(p, \delta) \cap \Omega_j)$ for those subdomains Ω_j that have nonempty intersection with $D(p, \delta)$. Below we will discuss the details of the interface singularities.

The discussion of the interface singularities for (2.2.8) are divided into two cases depending on whether the interface vertex p belongs to the boundary of Ω or the interior of Ω (See Figure 2.2).

Case 1. The interface vertex p belongs to the boundary of Ω. Let (λ_k, Θ_k), $\lambda_k = \sigma_k^2 > 0$, $k = 1, 2, 3, \cdots$ be the eigenvalues and eigenfunctions of the Sturm-Liouville problem around the interface vertex p:

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \quad \text{for } \omega_{j-1} \le \theta \le \omega_j \text{ and } 1 \le j \le J, \tag{2.2.9a}$$

$$\Theta'(0+) = \Theta'(\omega-) = 0, \tag{2.2.9b}$$

$$\Theta(\omega_j-) = \Theta(\omega_j+) \quad \text{for } 1 \le j \le J-1, \tag{2.2.9c}$$

$$\rho_j \Theta'(\omega_j-) = \rho_{j+1}\Theta'(\omega_j+) \quad \text{for } 1 \le j \le J-1, \tag{2.2.9d}$$

where the Θ_k's satisfy

$$\sum_{j=1}^{J} \int_{\omega_{j-1}}^{\omega_j} \Theta_i(\theta)\Theta_k(\theta)\rho_j d\theta = \delta_{ik}. \tag{2.2.10}$$

Moreover,

$$u - \sum_{\sigma_k<1} \kappa_k r^{\sigma_k}\Theta_k(\theta) \in H^2(\Omega_j \cap D(p,\delta)), \quad 1 \le j \le J, \tag{2.2.11}$$

where the κ_k's are called stress intensity factors, which can be computed by an extraction formula.

Case 2. The interface vertex p belongs to the interior of Ω. Let (λ_k, Θ_k), $\lambda_k = \sigma_k^2 > 0$, $k = 1, 2, 3, \cdots$ be the eigenvalues and eigenfunctions of the Sturm-Liouville problem around the interior vertex p:

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \quad \text{for } \omega_{j-1} \le \theta \le \omega_j \text{ and } 1 \le j \le J, \tag{2.2.12a}$$

$$\Theta(\omega_j-) = \Theta(\omega_j+) \quad \text{for } 1 \le j \le J-1, \tag{2.2.12b}$$

$$\rho_j \Theta'(\omega_j-) = \rho_{j+1}\Theta'(\omega_j+) \quad \text{for } 1 \le j \le J-1, \tag{2.2.12c}$$

$$\Theta(0+) = \Theta(2\pi-) = 0, \tag{2.2.12d}$$

$$\rho_1\Theta'(0+) = \rho_J\Theta'(2\pi-), \tag{2.2.12e}$$

where the Θ_k's satisfy

$$\sum_{j=1}^{J} \int_{\omega_{j-1}}^{\omega_j} \Theta_i(\theta)\Theta_k(\theta)\rho_j d\theta = \delta_{ik}. \tag{2.2.13}$$

Moreover,

$$u - \sum_{\sigma_k<1} \kappa_k r^{\sigma_k}\Theta_k(\theta) \in H^2(\Omega_j \cap D(p,\delta)), \quad 1 \le j \le J. \tag{2.2.14}$$

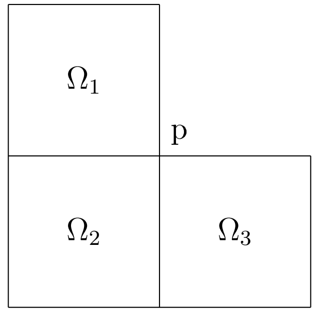

FIGURE 2.3. The domain Ω and its subdomains.

For the simplicity of presentation we will assume from here on there is only one interface vertex p of the subdomains near which u is singular. In this case, let ϱ_{cut} be a cut-off function which equals 1 in a neighborhood of the interface vertex p. Let s_l be defined by

$$s_l = r^{\sigma_l}\Theta_l(\theta)\varrho_{cut}, \tag{2.2.15}$$

where $\lambda_l = \sigma_l^2, l \geq 1$, are the eigenvalues of Sturm-Liouville problem at p and Θ_l's are the corresponding eigenfunctions. Let

$$u_S = \sum_{0<\sigma_l<1} \kappa_l s_l,$$

and $u_R = u - u_S$, then we have

$$u = u_S + u_R,$$

where $u_R|_{\Omega_j} \in H^2(\Omega_j)$. Moreover, we have the elliptic regularity estimate

$$\sum_{j=1}^{J} \|u_R\|_{H^2(\Omega_j)} + \sum_{0<\sigma_l<1} |\kappa_l| \leq C\|f\|_{L_2(\Omega)}. \tag{2.2.16}$$

Example 2.2.4. Let us consider the Sturm-Liouville problem on an L-shape domain with vertices $(0,0)$, $(0,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$, and $(0,1)$, which is partitioned into three squares Ω_1, Ω_2 and Ω_3 (See Figure 2.3). So the interface vertex is $(0,0)$ and $\omega_0 = \frac{\pi}{2}$, $\omega_1 = \pi$, $\omega_2 = \frac{3\pi}{2}$ and $\omega_3 = 2\pi$.

Suppose that $\rho_1 = \rho_3 = 50$, $\rho_2 = 1$. Then $\sigma_1 = 0.126276410744819...$ is the positive square root of the first or smallest eigenvalue of this Sturm-Liouville problem, and it is the only one which is less than one.

Actually, in this example, we can compute the first positive eigenvalue of the corresponding Sturm-Liouville problem which is between 0 and 1 by the following formula :

$$\sin(\frac{\sigma_1 \pi}{2}) = \sqrt{\frac{\rho_2(\rho_1 + \rho_2 + \rho_3)}{\rho_2(\rho_1 + \rho_2 + \rho_3) + \rho_1\rho_3}}, \tag{2.2.17}$$

where $\lambda_1 = \sigma_1^2$.

Using the equation (2.2.17) with $\rho_2 = 1$ and $\rho_1 = \rho_3$, we can construct a Sturm-Liouville problem whose first positive eigenvalue is as small as we want. If $\rho_1 = \rho_3 = 50$ and $\rho_2 = 1$, then $\sigma_1 = 0.126276410744819...$ If $\rho_1 = \rho_3 = 350$ and $\rho_2 = 1$, then $\sigma_1 = 0.048066746316346...$ If $\rho_1 = \rho_3 = 1300$ and $\rho_2 = 1$, then $\sigma_1 = 0.024962282535010...$

Remark 2.2.5. Note that if $\rho_1 = \rho_2 = \rho_3 = 1$, then $\sigma_1 = \frac{2}{3}$.

2.3 Extraction Formulas

From Subsection 2.2.1, we know that the solutions of Poisson problems with Dirichlet or Neumann boundary conditions on nonconvex domains have singular representations $u = \sum_{0<\sigma_l<1} \kappa_l s_l + u_R$, where $u_R \in H^2(\Omega)$ and s_l is determined by the interior angle of the corner. The same is true for the elliptic interface problem. The goal of this section is to develop formulas for computing the stress intensity factors κ_l in different cases. These formulas are called extraction formulas [5, 2, 30, 45, 61, 37, 50] and [41, 42, 32, 53, 54, 55].

2.3.1 Extraction Formulas for Poisson Problems

First let us consider the Poisson problem with the Dirichlet boundary condition on nonconvex domain. Suppose that Ω is a nonconvex polygonal domain and $f \in L_2(\Omega)$. Consider the following Dirichlet problem:

Find $u \in H_0^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in the domain } \Omega, \tag{2.3.1a}$$

$$u = 0 \quad \text{on the boundary } \partial\Omega. \tag{2.3.1b}$$

From Subsection 2.2.1, we know that

$$u = \sum_{\omega_l > \pi} \kappa_l s_l + u_R, \tag{2.3.2}$$

where $u_R \in H^2(\Omega)$ and s_l has the form

$$r^{\frac{\pi}{\omega_l}} \sin(\frac{\pi}{\omega_l}\theta)\varrho_l(r)$$

around the corner. Here ω_l is the interior angle of the corner where $\omega_l > \pi$, (r, θ) are the local polar coordinates around the corner, and ϱ_l is a cut-off function which equals 1 around the corner and 0 away from the corner.

In the following lemma, we derive the extraction formula for computing the stress intensity factor κ_l in the equation (2.3.2). First, let us define a related function.

Definition 2.3.1. *(Dual Singular Function)* Given the singular function $s = r^\sigma \sin(\sigma\theta)\varrho(r)$ around a corner of a polygonal domain, we call the function $s^* = r^{-\sigma} \sin(\sigma\theta)\varrho(r)$ the *dual singular function.*

Lemma 2.3.2. (Extraction Formula for κ_l) *Let $u \in H_0^1(\Omega)$ be the weak solution of the Poisson problem of (2.3.1). Then the stress intensity factors κ_l in the singular representation (2.3.2) can be computed by the extraction formula*

$$\kappa_l = \frac{1}{\pi} \int_\Omega (f s_l^* + u\Delta s_l^*) dx, \tag{2.3.3}$$

where

$$s_l = r^{\frac{\pi}{\omega_l}} \sin(\frac{\pi}{\omega_l}\theta)\varrho_l(r)$$

and

$$s_l^* = r^{-\frac{\pi}{\omega_l}} \sin(\frac{\pi}{\omega_l}\theta)\varrho_l(r).$$

Proof. Without loss of generality, we assume there is only one corner p_l with the interior angle $\omega_l > \pi$. Given any small $\delta > 0$, denote $D_\delta = B(p_l, \delta) \cap \Omega$, where $B(p_l, \delta) = \{x \in \mathbb{R}^2 : \|x - p_l\| < \delta\}$. Denote $\Omega_\delta = \Omega \setminus D_\delta$.

For a small δ, consider the integral

$$I_\delta = \int_{\Omega_\delta} (f s_l^* + u\Delta s_l^*)dx. \tag{2.3.4}$$

Using Lebesgue's dominated convergence theorem, we can show that

$$\lim_{\delta\to 0} I_\delta = \int_{\Omega} (f s_l^* + u\Delta s_l^*)dx. \tag{2.3.5}$$

We rewrite (2.3.4) as

$$I_\delta = I_1^\delta + \kappa_l I_2^\delta, \tag{2.3.6}$$

where $I_1^\delta = \int_{\Omega_\delta} (-(\Delta u_R)s_l^* + u_R\Delta s_l^*)dx$ and $I_2^\delta = \int_{\Omega_\delta} (-(\Delta s_l)s_l^* + s_l\Delta s_l^*)dx$.

After applying Green's formula and a direct computation, we have

$$\lim_{\delta\to 0} I_2^\delta = \pi. \tag{2.3.7}$$

So it remains to show that $\lim\limits_{\delta\to 0} I_1^\delta = 0$.

Applying Green's formula again, we have

$$I_1^\delta = \int_{\partial\Omega_\delta} (-\frac{\partial u_R}{\partial n} s_l^* + u_R \frac{\partial s_l^*}{\partial n})ds = I_3^\delta + I_4^\delta. \tag{2.3.8}$$

Here

$$I_3^\delta = \int_{\Gamma_\delta} (-\frac{\partial u_R}{\partial n} s_l^* + u_R \frac{\partial s_l^*}{\partial n})ds \tag{2.3.9}$$

and

$$I_4^\delta = \int_{\partial\Omega_\delta \setminus \Gamma_\delta} (-\frac{\partial u_R}{\partial n} s_l^* + u_R \frac{\partial s_l^*}{\partial n}) ds, \tag{2.3.10}$$

where $\Gamma_\delta = \partial B(p_l, \delta) \cap \bar{\Omega}$.

Note that $u_R = 0$ and $s_l^* = 0$ on the boundary $\partial\Omega$ and $\partial\Omega_\delta \setminus \Gamma_\delta \subset \partial\Omega$, so $I_4^\delta = 0$.

For sufficiently small δ , by a direct computation, we further simplify I_3^δ in the following way:

$$I_3^\delta = -\int_{\Gamma_\delta} (-\frac{\partial u_R}{\partial r} s_l^* + u_R \frac{\partial s_l^*}{\partial r}) ds = L_1 + L_2, \tag{2.3.11}$$

where

$$L_1 = \int_0^{\omega_l} \frac{\partial u_R}{\partial r} \sin(\frac{\pi}{\omega_l}\theta) \delta^{1-\frac{\pi}{\omega_l}} d\theta \tag{2.3.12}$$

and

$$L_2 = \frac{\pi}{\omega_l} \int_0^{\omega_l} u_R \sin(\frac{\pi}{\omega_l}\theta) \delta^{-\frac{\pi}{\omega_l}} d\theta. \tag{2.3.13}$$

Since $u_R \in H^2(\Omega)$, it follows from the Sobolev embedding theorem (cf. Theorem 2.1.9) that

$$u_R \in C^{0,\eta}(\bar{\Omega}) \quad \text{for any} \quad 0 < \eta < 1 \tag{2.3.14}$$

and hence there exists a positive constant C_η depending only on η and Ω such that

$$|u_R(x) - u_R(y)| \le C_\eta |x-y|^\eta \quad \text{for} \quad x, y \in \Omega, \tag{2.3.15}$$

which together with the fact that $u_R(p_l) = 0$ implies

$$|u_R(\delta, \theta)| \le C_\eta \delta^\eta. \tag{2.3.16}$$

It follows from (2.3.13) and (2.3.16) that, for some positive constant C_η' depending only on η and Ω,

$$|L_2| \le C_\eta' \delta^{\eta - \frac{\pi}{\omega_l}}, \tag{2.3.17}$$

which implies that, if we choose an η so that $\eta - \frac{\pi}{\omega_l} > 0$, then

$$\lim_{\delta \to 0} L_2 = 0. \tag{2.3.18}$$

Let $z = \frac{\partial u_R}{\partial r}$. Since $u_R \in H^2(\Omega)$, we have $z \in H^1(\Omega)$ and $z \in H^1(D_\delta)$. Let $\hat{z}(r,\theta) = z(\delta r, \theta)$, then $\hat{z} \in H^1(D_1)$.

By the Cauchy-Schwartz inequality, there exists a constant C depending only on ω_l such that

$$\begin{aligned} |\int_0^{\omega_l} \frac{\partial u_R}{\partial r} \sin(\frac{\pi}{\omega_l}\theta)d\theta| = |\int_0^{\omega_l} z(\delta,\theta)\sin(\frac{\pi}{\omega_l}\theta)d\theta| = |\int_0^{\omega_l} \hat{z}(1,\theta)\sin(\frac{\pi}{\omega_l}\theta)d\theta| \\ \leq C\|\hat{z}\|_{L_2(\Gamma_1)}, \end{aligned}$$

which together with Theorem 2.1.10 implies

$$|\int_0^{\omega_l} \frac{\partial u_R}{\partial r} \sin(\frac{\pi}{\omega_l}\theta)d\theta| \leq C\|\hat{z}\|_{H^1(D_1)}. \tag{2.3.19}$$

Since $\hat{z} \in H^1(D_1)$, by Theorem 2.1.9, we have $\hat{z} \in L_q(D_1)$ for any $q \geq 1$. Using Hölder's inequality, we obtain the estimate

$$\|\hat{z}\|_{L_2(D_1)} \leq C_q\|\hat{z}\|_{L_q(D_1)} \quad \text{for} \quad q > 2, \tag{2.3.20}$$

where C_q is a positive constant depending only on q and ω_l.

A direct computation implies

$$\begin{aligned} \|\hat{z}\|^q_{L_q(D_1)} &= \int_0^1 \int_0^{\omega_l} |\hat{z}(r,\theta)|^q r d\theta dr \\ &= \frac{1}{\delta^2}\int_0^\delta \int_0^{\omega_l} |z|^q r d\theta dr \\ &= \frac{1}{\delta^2}\|z\|^q_{L_q(D_\delta)} \end{aligned}$$

and hence

$$\|\hat{z}\|_{L_q(D_1)} = \delta^{-2/q}\|z\|_{L_q(D_\delta)}. \tag{2.3.21}$$

Similarly, we derive

$$|\hat{z}|_{H^1(D_1)} = |z|_{H^1(D_\delta)}. \tag{2.3.22}$$

Combining (2.3.12), (2.3.19)– (2.3.22), we obtain that, for any given $q > 2$,

$$|L_1| \leq C'_q(\delta^{1-\frac{\pi}{\omega_l}-\frac{2}{q}}\|z\|_{L_q(D_\delta)} + \delta^{1-\frac{\pi}{\omega_l}}|z|_{H^1(D_\delta)}), \tag{2.3.23}$$

where C_q' is a positive constant depending only on q and ω_l. It follows from Theorem 2.1.9 and (2.3.23) that, for a small δ,

$$|L_1| \le C_q' \delta^{1-\frac{\pi}{\omega_l}-\frac{2}{q}} \|z\|_{H^1(\Omega)}. \tag{2.3.24}$$

Choose q such that $\frac{2}{q} < 1 - \frac{\pi}{\omega_l}$, then (2.3.24) implies

$$\lim_{\delta \to 0} L_1 = 0, \tag{2.3.25}$$

which completes the proof. □

Next we consider the following Poisson problem with Neumann boundary condition:

Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in the domain } \Omega, \tag{2.3.26a}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on the boundary } \partial\Omega. \tag{2.3.26b}$$

From Subsection 2.2.1, we know that

$$u = \sum_{\omega_l > \pi} \kappa_l s_l + u_R, \tag{2.3.27}$$

where $u_R \in H^2(\Omega)$ and s_l has the form

$$r^{\frac{\pi}{\omega_l}} \cos(\frac{\pi}{\omega_l}\theta)\varrho_l(r)$$

around the corner. Here ω_l is the interior angle of the corner where $\omega_l > \pi$, (r, θ) are the local polar coordinates around the corner, and ϱ_l is the cut-off function which equals 1 around the corner and 0 away from the corner.

In this case, the corresponding dual singular functions are of the form

$$s^* = r^{-\sigma} \cos(\sigma\theta)\varrho(r)$$

around the particular corners of the polygonal domain. Then the extraction formulas for this problem are formulated in the following lemma.

Lemma 2.3.3. (Extraction Formula for κ_l) *Let $u \in H^1(\Omega)$ be the weak solution of the Poisson problem (2.3.26). Then the stress intensity factors κ_l in the singular representation (2.3.27) can be computed by the extraction formula*

$$\kappa_l = \frac{1}{\pi}\int_{\Omega}(fs_l^* + u\Delta s_l^*)dx, \tag{2.3.28}$$

where

$$s_l = r^{\frac{\pi}{\omega_l}}\cos(\frac{\pi}{\omega_l}\theta)\varrho_l(r),$$
$$s_l^* = r^{-\frac{\pi}{\omega_l}}\cos(\frac{\pi}{\omega_l}\theta)\varrho_l(r).$$

Proof. Without loss of generality, we assume there is only one corner p_l with the interior angle $\omega_l > \pi$. Given any small $\delta > 0$, denote $D_\delta = B(p_l, \delta) \cap \Omega$, where $B(p_l, \delta) = \{x \in \mathbb{R}^2 : \|x - p_l\| < \delta\}$. Denote $\Omega_\delta = \Omega \setminus D_\delta$.

For a small δ, consider the integral

$$I_\delta = \int_{\Omega_\delta}(fs_l^* + u\Delta s_l^*)dx. \tag{2.3.29}$$

Using Lebesgue's dominated convergence theorem, we can show that

$$\lim_{\delta\to 0} I_\delta = \int_{\Omega}(fs_l^* + u\Delta s_l^*)dx. \tag{2.3.30}$$

We rewrite (2.3.29) as

$$I_\delta = I_1^\delta + \kappa_l I_2^\delta, \tag{2.3.31}$$

where $I_1^\delta = \int_{\Omega_\delta}(-(\Delta u_R)s_l^* + u_R\Delta s_l^*)dx$ and $I_2^\delta = \int_{\Omega_\delta}(-(\Delta s_l)s_l^* + s_l\Delta s_l^*)dx$.

After applying Green's formula and a direct computation, we have

$$\lim_{\delta\to 0} I_2^\delta = \pi. \tag{2.3.32}$$

So it remains to show that $\lim_{\delta\to 0} I_1^\delta = 0$.

Applying Green's formula again, we have

$$I_1^\delta = \int_{\partial\Omega_\delta}(-\frac{\partial u_R}{\partial n}s_l^* + u_R\frac{\partial s_l^*}{\partial n})ds = I_3^\delta + I_4^\delta. \tag{2.3.33}$$

Here

$$I_3^\delta = \int_{\Gamma_\delta} (-\frac{\partial u_R}{\partial n} s_l^* + u_R \frac{\partial s_l^*}{\partial n}) ds \tag{2.3.34}$$

and

$$I_4^\delta = \int_{\partial\Omega_\delta \setminus \Gamma_\delta} (-\frac{\partial u_R}{\partial n} s_l^* + u_R \frac{\partial s_l^*}{\partial n}) ds, \tag{2.3.35}$$

where $\Gamma_\delta = \partial B(p_l, \delta) \cap \bar{\Omega}$.

Note that $\frac{\partial u_R}{\partial n} = 0$ and $\frac{\partial s_l^*}{\partial n} = 0$ on the boundary $\partial\Omega$ and $\partial\Omega_\delta \setminus \Gamma_\delta \subset \partial\Omega$, so $I_4^\delta = 0$.

For sufficiently small δ , by a direct computation, we further simplify I_3^δ in the following way:

$$I_3^\delta = -\int_{\Gamma_\delta} (-\frac{\partial u_R}{\partial r} s_l^* + u_R \frac{\partial s_l^*}{\partial r}) ds = L_1 + L_2, \tag{2.3.36}$$

where

$$L_1 = \int_0^{\omega_l} \frac{\partial u_R}{\partial r} \cos(\frac{\pi}{\omega_l}\theta) \delta^{1-\frac{\pi}{\omega_l}} d\theta \tag{2.3.37}$$

and

$$L_2 = \frac{\pi}{\omega_l} \int_0^{\omega_l} u_R \cos(\frac{\pi}{\omega_l}\theta) \delta^{-\frac{\pi}{\omega_l}} d\theta. \tag{2.3.38}$$

The regular part u_R at the vertex p_l may not be zero for the Neumann problem. But we can rewrite (2.3.38) as

$$L_2 = \frac{\pi}{\omega_l} \delta^{-\frac{\pi}{\omega_l}} \int_0^{\omega_l} \left[(u_R - u_R(p_l)) \cos(\frac{\pi}{\omega_l}\theta) + u_R(p_l) \cos(\frac{\pi}{\omega_l}\theta) \right] d\theta. \tag{2.3.39}$$

Since $\int_0^{\omega_l} \cos(\frac{\pi}{\omega_l}\theta) d\theta = 0$, we have

$$L_2 = \frac{\pi}{\omega_l} \delta^{-\frac{\pi}{\omega_l}} \int_0^{\omega_l} (\tilde{u}_R) \cos(\frac{\pi}{\omega_l}\theta) d\theta, \tag{2.3.40}$$

where $\tilde{u}_R = u_R - u_R(p_l)$.

Now we can repeat the argument in the proof of Lemma 2.3.2 to prove $\lim_{\delta\to 0} L_1 = 0$ and $\lim_{\delta\to 0} L_2 = 0$. $\square$

2.3.2 Extraction Formulas for Elliptic Interface Problems

In addition to the assumptions for the domain Ω in Subsection 2.2.2 we further assume for simplicity the domain Ω has only one interface vertex p. From Subsection 2.2.2, we have, for the solution of the interface problem (2.2.8), the following singular function representation

$$u = \sum_{0<\sigma_l<1} \kappa_l s_l + u_R, \tag{2.3.41}$$

where $u_R \in H^2(\Omega_j), 1 \leq j \leq J$ and s_l has the form $r^{\sigma_l}\Theta_j(\theta)\varrho_l(r)$ around the interface vertex p. Here $\{\sigma_l^2, \Theta_l\}, 0 < \sigma_l < 1$ are the first few eigenvalues and eigenfunctions of the corresponding Sturm-Liouville problem (2.2.9) or (2.2.12), (r, θ) are the local polar coordinates around the interface vertex p, and ϱ_l is the cut-off function which equals 1 around the interface vertex p and 0 away from p.

In this case, the dual singular function of $s_l = r^{\sigma_l}\Theta_l(\theta)\varrho_l(r)$ is defined by $s_l^* = r^{-\sigma_l}\Theta_l(\theta)\varrho_l(r)$. We have the following lemma on the extraction formula for the stress intensity factors of the interface problem (2.2.8).

Lemma 2.3.4. (Extraction Formula for κ_l) *Let $u \in H^1(\Omega)$ be the weak solution of the elliptic interface problem (2.2.8). Then the stress intensity factors κ_l in (2.3.41) can be computed by the extraction formula*

$$\kappa_l = \frac{1}{2\sigma_l}\int_\Omega (f s_l^* + \rho u \Delta s_l^*)dx. \tag{2.3.42}$$

Proof. Given any small $\delta > 0$, denote $D_\delta = B(p, \delta) \cap \Omega$, where $B(p, \delta) = \{x \in \mathbb{R}^2 : \|x - p\| < \delta\}$. Denote $\Omega_\delta = \Omega \setminus D_\delta$ and $\Omega_{\delta,j} = \Omega_\delta \cap \Omega_j$.

For a small δ, consider the integral

$$I_\delta = \int_{\Omega_\delta} (f s_l^* + \rho u \Delta s_l^*)dx. \tag{2.3.43}$$

Using Lebesgue's dominated convergence theorem, we can show that

$$\lim_{\delta\to 0} I_\delta = \int_\Omega (f s_l^* + \rho u \Delta s_l^*)dx. \tag{2.3.44}$$

We rewrite (2.3.43) as

$$I_\delta = I_1^\delta + I_2^\delta, \tag{2.3.45}$$

where $I_1^\delta = \int_{\Omega_\delta}(-\rho(\Delta u_R)s_l^* + \rho u_R \Delta s_l^*)dx$ and $I_2^\delta = \sum_{0<\sigma_{l'}<1} \kappa_{l'} \int_{\Omega_\delta}(-\rho(\Delta s_{l'})s_l^* + \rho s_{l'}\Delta s_l^*)dx$.

Applying Green's formula on each subdomain of Ω_δ and using the fact that $\frac{\partial s_l}{\partial n} = 0$ and $\frac{\partial s_l^*}{\partial n} = 0$ on the boundary $\partial\Omega$, we have

$$I_2^\delta = \sum_{0<\sigma_{l'}<1} -\kappa_{l'} \int_{\partial D_\delta} (-\rho \frac{\partial s_{l'}}{\partial r} s_l^* + \rho s_{l'} \frac{\partial s_{l'}^*}{\partial r}) ds. \tag{2.3.46}$$

When δ is small, (2.2.9)-(2.2.10) (or (2.2.12)-(2.2.13)) and the definition of s_l and s_l^* imply

$$\begin{aligned}\int_{\partial D_\delta} (-\rho \frac{\partial s_{l'}}{\partial r} s_l^* + \rho s_{l'} \frac{\partial s_l^*}{\partial r}) ds &= \sum_{j=1}^{J} \int_{\theta_{j-1}}^{\theta_j} (-\rho_j \sigma_{l'} \delta^{\sigma_{l'}-\sigma_l} \Theta_{l'} \Theta_l \\ &\quad + \rho_j(-\sigma_l)\delta^{\sigma_{l'}-\sigma_l}\Theta_{l'}\Theta_l)d\theta \\ &= -\delta^{l'-l}(\sigma_l + \sigma_{l'})\delta_{l'l} \end{aligned} \tag{2.3.47}$$

and hence $I_2^\delta = 2\kappa_l \sigma_l$. So it remains to show that $\lim_{\delta\to 0} I_1^\delta = 0$.

Applying Green's formula on $\Omega_{\delta,j}$ for $1 \le j \le J$, we have

$$I_1^\delta = \sum_{j=1}^{J} \int_{\partial\Omega_{\delta,j}} (-\rho_j \frac{\partial u_R}{\partial n} s_l^* + \rho_j u_R \frac{\partial s_l^*}{\partial n}) ds = I_3^\delta + I_4^\delta. \tag{2.3.48}$$

Here

$$I_3^\delta = \int_{\Gamma_\delta} (-\rho \frac{\partial u_R}{\partial n} s_l^* + \rho u_R \frac{\partial s_l^*}{\partial n}) ds \tag{2.3.49}$$

and

$$I_4^\delta = \sum_{j=1}^{J} \int_{\partial\Omega_{\delta,j}\setminus\Gamma_\delta} (-\rho \frac{\partial u_R}{\partial n} s_l^* + \rho u_R \frac{\partial s_l^*}{\partial n}) ds, \tag{2.3.50}$$

where $\Gamma_\delta = \partial B(p,\delta) \cap \bar{\Omega}$.

Since $\frac{\partial u_R}{\partial n}$ and $\frac{\partial s_l^*}{\partial n}$ are zero on the boundary $\partial\Omega$, $\left[\rho \frac{\partial u_R}{\partial n}\right]$ and $\left[\rho \frac{\partial s_l^*}{\partial n}\right]$ are zero on the interface Γ, it follows that $I_4^\delta = 0$.

For sufficiently small δ, we further simplify I_3^δ in the following way:

$$I_3^\delta = -\int_{\Gamma_\delta} (-\rho \frac{\partial u_R}{\partial r} s_l^* + \rho u_R \frac{\partial s_l^*}{\partial r}) ds = L_1 + L_2, \tag{2.3.51}$$

where

$$L_1 = \sum_{j=1}^{J} \int_{\theta_{j-1}}^{\theta_j} \rho_j \frac{\partial u_R}{\partial r} \Theta_l \delta^{1-\sigma_l} d\theta \tag{2.3.52}$$

and

$$L_2 = \sigma_l \sum_{j=1}^{J} \int_{\theta_{j-1}}^{\theta_j} \rho_j u_R \Theta_l \delta^{-\sigma_l} d\theta. \tag{2.3.53}$$

The regular part u_R at the interface vertex p may not be zero, but we can rewrite (2.3.53) as

$$L_2 = \sigma_l \sum_{j=1}^{J} \int_{\theta_{j-1}}^{\theta_j} \left[\rho_j (u_R - u_R(p)) \Theta_l \delta^{-\sigma_l} + \rho_j u_R(p) \Theta_l \delta^{-\sigma_l}\right] d\theta. \tag{2.3.54}$$

Since $\Theta_l''(\theta) = -\sigma_l^2 \Theta_l(\theta) \quad \text{for} \quad \theta_{j-1} < \theta < \theta_j$ and $1 \le j \le J$, we have

$$\sigma_l \sum_{j=1}^{J} \int_{\theta_{j-1}}^{\theta_j} \rho_j u_R(p) \Theta_l \delta^{-\sigma_l} d\theta = -\frac{1}{\sigma_l} \delta^{-\sigma_l} u_R(p) \sum_{j=1}^{J} \int_{\theta_{j-1}}^{\theta_j} \rho_j \Theta_l'' d\theta \tag{2.3.55}$$

which, together with the conditions (2.2.9b) and (2.2.9d) (or (2.2.12c) and (2.2.12e)), implies

$$\sigma_l \sum_{j=1}^{J} \int_{\theta_{j-1}}^{\theta_j} \rho_j u_R(p) \Theta_l \delta^{-\sigma_l} d\theta = 0. \tag{2.3.56}$$

Therefore,

$$L_2 = \sigma_l \sum_{j=1}^{J} \int_{\theta_{j-1}}^{\theta_j} \rho_j \tilde{u}_R \Theta_l \delta^{-\sigma_l} d\theta, \tag{2.3.57}$$

where $\tilde{u}_R = u_R - u_R(p)$.

Denote $L_{1,j} = \int_{\theta_{j-1}}^{\theta_j} \rho_j \frac{\partial u_R}{\partial r} \Theta_l \delta^{1-\sigma_l} d\theta$ and $L_{2,j} = \sigma_l \int_{\theta_{j-1}}^{\theta_j} \rho_j \tilde{u}_R \Theta_l \delta^{-\sigma_l} d\theta$.

Since $\tilde{u}_R \in H^2(\Omega_j)$ (cf. Subsection 2.2.2) and $\frac{\partial \tilde{u}_R}{\partial r} = \frac{\partial u_R}{\partial r} \in H^1(\Omega_j)$, we can repeat the argument in the proof of Lemma 2.3.2 to prove $\lim_{\delta \to 0} L_{1,j} = 0$ and $\lim_{\delta \to 0} L_{2,j} = 0$ for $1 \le j \le J$. Hence $\lim_{\delta \to 0} L_1 = 0$ and $\lim_{\delta \to 0} L_2 = 0$. $\square$

It is not difficult to extend the extraction formula in Lemma 2.3.4 to the nonhomogeneous interface problem with the Neumann boundary condition.

In addition to the assumptions for the problem (2.2.7), we further assume there are functions g_Γ and h_Γ on Γ such that there exists a function G on $\bar{\Omega}$ which satisfies

$$\begin{aligned} G|_{\Omega_j} &\in H^2(\Omega_j) \quad \text{for} \quad 1 \le j \le J, \\ \rho \frac{\partial G}{\partial n} &= g_\Gamma \quad \text{on the boundary } \partial\Omega, \\ [G] &= 0 \quad \text{on the interface } \Gamma, \end{aligned}$$

and

$$\left[\rho \frac{\partial G}{\partial n} \right] = h_\Gamma \quad \text{on the interface } \Gamma.$$

Now we consider the nonhomogeneous interface problem with the Neumann boundary condition:

Find $u \in H^1(\Omega)$ such that

$$-\rho_j \Delta u = f \quad \text{in } \Omega_j, 1 \le j \le J, \tag{2.3.58a}$$

$$\rho \frac{\partial u}{\partial n} = g_\Gamma \quad \text{on the boundary } \partial\Omega, \tag{2.3.58b}$$

$$[u] = 0 \quad \text{on the interface } \Gamma, \tag{2.3.58c}$$

$$\left[\rho \frac{\partial u}{\partial n} \right] = h_\Gamma \quad \text{on the interface } \Gamma. \tag{2.3.58d}$$

By our assumptions, $\tilde{u} = u - G$ is the solution of the following homogeneous interface problem with Neumann boundary condition:

$$-\rho_j \Delta \tilde{u} = f + \rho_j \Delta G \text{ in } \Omega_j, 1 \le j \le J, \tag{2.3.59a}$$

$$\rho \frac{\partial \tilde{u}}{\partial n} = 0 \text{ on the boundary } \partial\Omega, \tag{2.3.59b}$$

$$[\tilde{u}] = 0 \text{ on the interface } \Gamma, \tag{2.3.59c}$$

$$\left[\rho \frac{\partial \tilde{u}}{\partial n} \right] = 0 \text{ on the interface } \Gamma. \tag{2.3.59d}$$

So Lemma 2.3.4 implies that the stress intensity factors κ_l for the problem (2.3.58) can be computed by

$$\kappa_l = \frac{1}{2\sigma_l} \int_{\Omega} ((f + \rho\Delta G)s_l^* + \rho(u - G)\Delta s_l^*)dx. \tag{2.3.60}$$

By Green's formula and a similar argument in the proof of Lemma 2.3.3, (2.3.60) implies that

$$\kappa_l = \frac{1}{2\sigma_l} \left[\int_{\Omega} (f s_l^* + \rho u \Delta s_l^*)dx + \int_{\partial\Omega} \rho h_\Gamma s_l^* ds + \int_{\Gamma} \rho g_\Gamma s_l^* ds \right]. \tag{2.3.61}$$

In summary, we have the following lemma.

Lemma 2.3.5. (Extraction Formula for κ_l of the Nonhomogeneous Interface Problem with the Neumann Boundary Condition) *Let $u \in H^1(\Omega)$ be the weak solution of the elliptic interface problem (2.3.58). Then $\tilde{u} = u - G$ has the singular function representation (2.3.41) and the stress intensity factors κ_l in (2.3.41) can be computed by the extraction formula (2.3.61).*

2.4 Finite Element Methods

In this section we discuss some basic facts about finite element methods. The basic references are [20, 25].

We will use the Poisson problem with the Dirichlet boundary condition as a model problem.

Suppose that Ω is a polygonal domain and $f \in L_2(\Omega)$. Consider the following Dirichlet problem:

Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in the domain } \Omega, \tag{2.4.1a}$$

$$u = 0 \quad \text{on the boundary } \partial\Omega. \tag{2.4.1b}$$

Its weak formulation is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \tag{2.4.2}$$

where

$$a(u, v) = \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v dx.$$

The well-posedness of the problem (2.4.2) is guaranteed by the following Lax-Milgram theorem (cf. [33]).

Theorem 2.4.1. *Let H be a Hilbert space with inner product $(\cdot, \cdot)$, and let $B(u, v)$ be a bilinear form on $H \times H, u \in H, v \in H$ such that*

$$|B(u, v)| \leq C_1 \|u\|_H \|v\|_H, \tag{2.4.3a}$$

$$|B(u, u)| \geq C_2 \|u\|_H^2, \tag{2.4.3b}$$

with $C_1 > 0$, $C_2 > 0$.

Let $f \in H'$, i.e., f is a bounded linear functional on H. Then there exists a unique $u_0 \in H$ such that

$$B(u_0, v) = f(v) \quad \forall v \in H. \tag{2.4.4}$$

We want to construct a finite dimensional subspace of $H_0^1(\Omega)$ and solve the equation (2.4.2) on that subspace. This can be carried out by a conforming finite element method. Now we introduce the basic terminology of this method.

Definition 2.4.2. *(Triangulation)* Let Ω be a polygonal domain. A triangulation $\mathscr{T}_h$ of Ω is a subdivision consisting of triangles with the property that no vertex of any triangle lies in the interior of an edge of another triangle. Denote $h = \max_{T \in \mathscr{T}_h} \operatorname{diam} T$.

Definition 2.4.3. *(Quasi-uniform)* If a family $\{\mathscr{T}_h\}$ of triangulations of Ω satisfies the following condition: there exists a positive constant C such that

$$\min\{\operatorname{diam} B_T : T \in \mathscr{T}_h\} \geq Ch \tag{2.4.5}$$

for all h, where B_T is the largest ball inscribed in T and $h = \max_{T\in\mathscr{T}_h} \operatorname{diam} T$, then we say this family is quasi-uniform.

Definition 2.4.4. *(P_1 Finite Element Space on $\mathscr{T}_h$)* Let V_h be the space of continuous piecewise P_1 polynomials on the triangulation $\mathscr{T}_h$, or,

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_T \text{ is a first order polynomial for any } T \in \mathscr{T}_h\}.$$

Let $\mathring{V}_h$ be the subspace of V_h defined by

$$\mathring{V}_h = \{v \in V_h : v|_{\partial\Omega} = 0\}$$

or

$$\mathring{V}_h = V_h \cap H_0^1(\Omega).$$

Remark 2.4.5. $\mathring{V}_h$ is a subspace of the space $H_0^1(\Omega)$, so we refer to the corresponding finite element method as a conforming finite element method.

Once the finite element space is chosen ($\mathring{V}_h$ in our case), the discrete version of the weak problem (2.4.2) is:
Find $u_h \in \mathring{V}_h$ such that

$$a(u_h, v) = (f, v) \text{ for all } v \in \mathring{V}_h. \tag{2.4.6}$$

Because of the Lax-Milgram Theorem 2.4.1, the well-posedness of the equation (2.4.6) can be easily verified.

From (2.4.2) and (2.4.6), we have the following Galerkin orthogonality ([20, Proposition 2.5.9]) for $u - u_h$:

$$a(u - u_h, v) = 0 \quad \text{for} \quad v \in \mathring{V}_h. \tag{2.4.7}$$

Next, we consider the error between the two functions u and u_h. First, we consider Céa's lemma ([20, Theorem 2.8.1]), which is a consequence of (2.4.7). It shows that the approximation u_h to u is quasi-optimal.

Lemma 2.4.6. (Céa's Lemma) *Suppose u and u_h are the solutions of (2.4.2) and (2.4.6). Then there exists a positive constant C independent of the subspace $\mathring{V}_h$ such that*

$$\|u - u_h\|_{H^1(\Omega)} \leq C \inf_{v \in \mathring{V}_h} \|u - v\|_{H^1(\Omega)}.$$

Because of Céa's lemma, we can focus on finding a specific function $v \in V_h$ where $\|u - v\|_{H^1(\Omega)}$ can be estimated in terms of h. That specific function is the interpolant of u on V_h. Let $\Pi_h : \mathcal{C}^0(\bar{\Omega}) \to V_h$ be the nodal interpolant operator defined by

$$\Pi_h v = v \qquad \text{at all the vertices of } \mathcal{T}_h.$$

Then we have the following interpolation error estimate. In the case of a convex polygon, this is a standard result. A proof of the general case is given in Appendix A.

Theorem 2.4.7. (Interpolation Error Estimate) *Let u be the solution of (2.4.2). Let $c_1, c_2, \cdots, c_{N_\Omega}$ be the corners of Ω and $\omega_1, \omega_2, \cdots, \omega_{N_\Omega}$ be the interior angle of the corners. Let $\omega = \max\{\omega_1, \omega_2, \cdots, \omega_{N_\Omega}\}$ and*

$$\beta = \max\{1, \frac{\pi}{\omega}\}.$$

If $\beta = 1$, we have

$$\|u - \Pi_h u\|_{L_2(\Omega)} + h|u - \Pi_h u|_{H^1(\Omega)} \leq Ch^2 \|u\|_{H^2(\Omega)}, \tag{2.4.8}$$

where the constant C is independent of the mesh size h. If $\beta < 1$, we have

$$\|u - \Pi_h u\|_{L_2(\Omega)} + h|u - \Pi_h u|_{H^1(\Omega)} \leq Ch^{1+\beta}(\|u_R\|_{H^2(\Omega)} + \sum_{\omega_i > \pi} |\kappa_i|), \tag{2.4.9}$$

where the constant C is independent of the mesh size h.

Remark 2.4.8. Based on the regularity result for u (Theorem 2.2.1), we know that $u \in \mathcal{C}^0(\bar{\Omega})$ and hence $\Pi_h u$ is well-defined.

Applying Theorem 2.2.1, Lemma 2.4.6 and Theorem 2.4.7, we have the following error estimate.

Theorem 2.4.9. (H^1 Error Estimate for u_h) *Let u be the solution of (2.4.2) and u_h be the solution of (2.4.6). Let $c_1, c_2, \cdots, c_{N_\Omega}$ be the corners of Ω and $\omega_1, \omega_2, \cdots, \omega_{N_\Omega}$ be the interior angle of the corners. Let $\omega = \max\{\omega_1, \omega_2, \cdots, \omega_{N_\Omega}\}$ and*

$$\beta = \max\{1, \frac{\pi}{\omega}\}.$$

Then we have

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^\beta \|f\|_{L_2(\Omega)}, \tag{2.4.10}$$

where the constant C is independent of the mesh size h.

Remark 2.4.10. The above discussion can be generalized to the elliptic interface problem.

We now consider the error estimate for $u - u_h$ in the L_2 norm. To estimate $\|u - u_h\|_{L_2(\Omega)}$, we use a duality argument. Let w be the solution of

$$-\Delta w = e \quad \text{in} \quad \Omega, \tag{2.4.11a}$$

$$w = 0 \quad \text{on the boundary} \quad \partial\Omega, \tag{2.4.11b}$$

where $e = u - u_h$. The variational formulation of this problem is: find $w \in H_0^1(\Omega)$ such that

$$a(w, v) = (e, v) \quad \forall v \in H_0^1(\Omega). \tag{2.4.12}$$

Therefore

$$\|u - u_h\|^2_{L_2(\Omega)} = (u - u_h, u - u_h)$$

$$= a(w, u - u_h)$$
$$= a(w - \Pi_h w, u - u_h) \quad \text{(by Galerkin orthogonality (2.4.7))}$$
$$\leq C\|w - \Pi_h w\|_{H^1(\Omega)}\|u - u_h\|_{H^1(\Omega)}$$
$$\leq Ch^\beta\|u - u_h\|_{H^1(\Omega)}\|e\|_{L_2(\Omega)} \quad \text{(by (2.2.3) and (2.4.8))}.$$

By Theorem 2.4.9, we have $\|u - u_h\|_{H^1(\Omega)} \leq Ch^\beta\|f\|_{L_2(\Omega)}$. Therefore,

$$\|u - u_h\|_{L_2(\Omega)} \leq Ch^{2\beta}\|f\|_{L_2(\Omega)}.$$

Thus we have proved the following theorem.

Theorem 2.4.11. (L_2 Error Estimate for u_h) *Let u be the solution of (2.4.2) and u_h be the solution of (2.4.6). Let $c_1, c_2, \cdots, c_{N_\Omega}$ be the corners of Ω and $\omega_1, \omega_2, \cdots, \omega_{N_\Omega}$ be the interior angle of the corners. Let $\omega = \max\{\omega_1, \omega_2, \cdots, \omega_{N_\Omega}\}$ and*

$$\beta = \max\{1, \frac{\pi}{\omega}\}.$$

Then we have

$$\|u - u_h\|_{L_2(\Omega)} \leq Ch^{2\beta}\|f\|_{L_2(\Omega)}, \tag{2.4.13}$$

where the constant C is independent of the mesh size h.

2.5 Multigrid Methods

In this section we discuss multigrid methods. The main references are [38, 10, 64, 11, 24, 20].

We consider the model problem (2.4.2). To approximate the solution u, we construct a nested sequence of triangulations $\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_k, \cdots$ over the polygonal domain Ω by the following procedure. Suppose that $\mathscr{T}_1$ is given, then the triangulation $\mathscr{T}_k$ is obtained by connecting the midpoints of the edges of the triangles of the coarser triangulation $\mathscr{T}_{k-1}$ for $k > 1$. On the triangulation $\mathscr{T}_k$ we define the finite element space $V_k \subset V = H_0^1(\Omega)$:

$$V_k = \{v \in C^0(\bar{\Omega}) : v|_T \text{ is a first order polynomial} \quad \forall T \in \mathscr{T}_k\} \cap H_0^1(\Omega).$$

It is easy to see that $V_{k-1} \subset V_k$.

First we introduce the basic terminology for multigrid methods.

We define a mesh-dependent inner product $(\cdot, \cdot)_k$ on V_k by

$$(v, w)_k = h_k^2 \sum_{i=1}^{n_k} v(p_i) w(p_i),$$

where $\{p_i\}_{i=1}^{n_k}$ is the set of internal vertices of $\mathscr{T}_k$.

The linear operators $A_k : V_k \to V_k$ are defined by

$$(A_k v, w)_k = a(v, w) \quad \forall v, w \in V_k.$$

The operators $Q_k : L_2(\Omega) \to V_k$ are defined by

$$(Q_k u, v)_k = (u, v) \quad \forall u \in L_2(\Omega), v \in V_k.$$

The discrete weak problem (2.4.6) is then equivalent to

$$A_k u_k = F_k, \tag{2.5.1}$$

where $F_k = Q_k f$.

The coarse-to-fine operator $I_{k-1}^k : V_{k-1} \to V_k$ is defined to be the natural injection, or equivalently,

$$I_{k-1}^k v = v \quad \forall v \in V_{k-1}.$$

The fine-to-coarse operator $I_k^{k-1} : V_k \to V_{k-1}$ is defined to be the transpose of I_{k-1}^k, or equivalently,

$$(I_k^{k-1} v, w)_{k-1} = (v, I_{k-1}^k w)_k \quad \forall v \in V_k, v \in V_{k-1}.$$

Algorithm 2.5.1. (The k^{th} Level Iteration) *$MG(k, z_0, F_k)$ is the approximate solution of the equation*

$$A_k z = F_k$$

obtained by the k^{th} level iteration with initial guess z_0. Let $R_k : V_k \to V_k$ be an approximation of A_k^{-1} and $R_1 = A_1^{-1}$.

For $k = 1$, $MG(1, z_0, F_1)$ is the solution obtained from a direct method. In other words,

$$MG(1, z_0, g) = R_1 F_1.$$

For $k > 1$, $MG(k, z_0, F_k)$ is obtained recursively in three steps.

Presmoothing Step. For $1 \leq l \leq m_1$, let

$$z_l = z_{l-1} + R_k(F_k - A_k z_{l-1}).$$

Error Correction Step. Let $\bar{F}_{k-1} := I_k^{k-1}(F_k - A_k z_{m_1})$ and $q_0 = 0$. For $1 \leq i \leq p$, let

$$q_i = MG(k-1, q_{i-1}, \bar{F}_{k-1}).$$

Then we define

$$z_{m_1+1} = z_{m_1} + I_{k-1}^k q_p.$$

Postsmoothing Step. For $m_1 + 2 \leq l \leq m_1 + m_2 + 1$, let

$$z_l = z_{l-1} + R_k(F_k - A_k z_{l-1}).$$

Then the final output of the k^{th} level iteration is

$$MG(k, z_0, F_k) = z_{m_1+m_2+1}.$$

Here $p = 1$ or $p = 2$. When $p = 1$ it is called a V-cycle method. When$p = 2$ it is called a W-cycle method.

When applying the k^{th} level iteration to (2.5.1), we use the following approach. We take the initial guess to be $I_{k-1}^k \hat{u}_{k-1}$, where $\hat{u}_{k-1}$ is the approximate solution already obtained for the equation $A_{k-1}u_{k-1} = F_{k-1}$. Then we apply the k^{th} level iteration r times.

Algorithm 2.5.2. (The Full Multigrid Algorithm) *For $k = 1$, $\hat{u}_1 = R_1 F_1$.*

For $k \geq 2$, the approximate solution $\hat{u}_k$ is obtained recursively from

$$
\begin{aligned}
u_0^k &= I_{k-1}^k \hat{u}_{k-1}, \\
u_l^k &= MG(k, u_{l-1}^k, F_k), \quad 1 \leq l \leq r, \\
\hat{u}_k &= u_r^k.
\end{aligned}
$$

For simplicity, we consider the convergence of the one-sided W-cycle method, i.e., $p = 2$, $m_1 = m$ and $m_2 = 0$ in the algorithm 2.5.1. Then we have the following convergence result (cf. [20, Theorem 6.5.9]).

Theorem 2.5.3. (Convergence of the k^{th} Level Iteration for the One-sided W-Cycle) *For any $0 < \gamma < 1$, m can be chosen large enough such that*

$$\|z - MG(k, z_0, F_k)\|_{H^1(\Omega)} \leq \gamma \|z - z_0\|_{H^1(\Omega)}, \quad \text{for } k = 1, 2, \ldots.$$

Remark 2.5.4. Similar convergence results also hold for other W-cycle methods.

The convergence of the full multigrid method is a consequence of the convergence of k^{th} level iteration (cf. [20, Theorem 6.7.1])

Theorem 2.5.5. (Full Multigrid Convergence) *If the k^{th} level iteration is a contraction with a contraction number γ independent of k and if r is large enough, then there exists a constant $C > 0$ such that*

$$\|u_k - \hat{u}_k\|_{H^1(\Omega)} \leq C h_k^{\beta} |u|_{H^{1+\beta}(\Omega)},$$

where $\beta = \max\{1, \frac{\pi}{\omega}\}$.

Chapter 3

Vector Function Spaces and Helmholtz/Hodge Decompositions

The numerical methods we develop in this dissertation are based on the Helmholtz/ Hodge decomposition for vector fields. Since Maxwell's equations involve the divergence operator and curl operator, it is easy to see that Helmholtz/Hodge decomposition can play a role in the study of Maxwell's equation. In this chapter, we review the Helmholtz/Hodge decomposition for two-dimensional vector fields, since we will focus on the two-dimensional Maxwell's equations. The main references are [14, 35, 49].

3.1 Definitions and Properties of the Vector Function Spaces $H(div;\Omega)$ and $H(curl;\Omega)$

Let $\Omega \subset \mathbb{R}^2$ be an open set with a Lipschitz boundary Γ. The vector function spaces naturally related to the variational formulation of Maxwell's equations are $H(div,\Omega)$ and $H(curl,\Omega)$.

Definition 3.1.1. Let $\boldsymbol{u} = (u_1(x_1,x_2), u_2(x_1,x_2))$ belong to $[L_2(\Omega)]^2$. We say that $\boldsymbol{\nabla}\cdot\boldsymbol{u} \in L_2(\Omega)$ if there exists a function $v \in L_2(\Omega)$ such that

$$(v,\phi) = -(\boldsymbol{u},\boldsymbol{\nabla}\phi) \quad \forall \phi \in \mathcal{D}(\Omega).$$

We will then take $\nabla\cdot\boldsymbol{u}$ top be v. The vector function space $H(div;\Omega)$ is defined as follows:

$$H(div;\Omega) = \{\boldsymbol{u} \in [L_2(\Omega)]^2 : \boldsymbol{\nabla}\cdot\boldsymbol{u} \in L_2(\Omega)\},$$

with the norm

$$\|\boldsymbol{u}\|_{H(div;\Omega)} = \{\|\boldsymbol{u}\|^2_{L_2(\Omega)} + \|\boldsymbol{\nabla}\cdot\boldsymbol{u}\|^2_{L_2(\Omega)}\}^{1/2}.$$

Definition 3.1.2. Let $\boldsymbol{u} = (u_1(x_1, x_2), u_2(x_1, x_2))$ belong to $[L_2(\Omega)]^2$. We say that $\boldsymbol{\nabla} \times \boldsymbol{u} \in L_2(\Omega)$ if there exists a function $v \in L_2(\Omega)$ such that

$$(v, \phi) = (\boldsymbol{u}, \boldsymbol{\nabla} \times \phi) \quad \forall \phi \in \mathcal{D}(\Omega),$$

where $\boldsymbol{\nabla} \times \phi = (\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1})$. We then define $\nabla \times \boldsymbol{u}$ to be v. The vector function space $H(curl; \Omega)$ is defined as follows:

$$H(curl; \Omega) = \{\boldsymbol{u} \in [L_2(\Omega)]^2 : \boldsymbol{\nabla} \times \boldsymbol{u} \in L_2(\Omega)\},$$

with the norm

$$\|\boldsymbol{u}\|_{H(curl;\Omega)} = \{\|\boldsymbol{u}\|^2_{L_2(\Omega)} + \|\boldsymbol{\nabla} \times \boldsymbol{u}\|^2_{L_2(\Omega)}\}^{1/2}.$$

First we discuss the properties of the spaces $H(div; \Omega)$.

Theorem 3.1.3. (cf. [35, Theorem 2.4.]) *The space $\mathcal{D}(\bar{\Omega})^2$ is dense in $H(div; \Omega)$.*

Because of the following theorem (cf. [35, Theorem 2.5]), the normal trace of $H(div; \Omega)$ can be defined.

Theorem 3.1.4. *The mapping $\gamma_n : \boldsymbol{v} \to \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma$ defined on $\mathcal{D}(\bar{\Omega})^2$ can be extended by continuity to a continuous linear mapping, still denoted by γ_n, from $H(div; \Omega)$ into $H^{-1/2}(\Gamma)$, the dual space of $H^{1/2}(\Gamma)$.*

We call $\gamma_n \boldsymbol{v}$ the normal trace of $\boldsymbol{v}$ on Γ and it is denoted by $\boldsymbol{v} \cdot \boldsymbol{n}$. We also denote $(\gamma_n \boldsymbol{v})(\phi)$ for any $\phi \in H^{1/2}(\Gamma)$ and $\gamma_n \boldsymbol{v} \in H^{-1/2}(\Gamma)$ by $< \boldsymbol{v} \cdot \boldsymbol{n}, \phi >_\Gamma$.

Because of Theorem 3.1.4, we have a generalized versions of Green's formulas ([35, Corollary 2.6]).

Corollary 3.1.5. *Let $\boldsymbol{v} \in H(div; \Omega)$ and $\phi \in H^1(\Omega)$, then*

$$(v, \boldsymbol{\nabla} \phi) + (\boldsymbol{\nabla} \cdot \boldsymbol{v}, \phi) =< \boldsymbol{v} \cdot \boldsymbol{n}, \phi >_\Gamma . \tag{3.1.1}$$

If $u \in H^1(\Omega)$ and $\Delta u \in L_2(\Omega)$, then $\boldsymbol{\nabla} u$ is in $H(div;\Omega)$. Therefore Corollary 3.1.5 implies the following corollary.

Corollary 3.1.6. *Let $u \in H^1(\Omega)$ and $\Delta u \in L_2(\Omega)$. Then $\frac{\partial u}{\partial n} \in H^{-1/2}(\Gamma)$ and*

$$(\boldsymbol{\nabla} u, \boldsymbol{\nabla} v) + (\Delta u, v) =< \frac{\partial u}{\partial n}, v >_\Gamma \quad \forall v \in H^1(\Omega).$$

Definition 3.1.7. Let $H_0(div;\Omega)$ denote

$$Ker(\gamma_n) = \{\boldsymbol{u} \in H(div;\Omega) : \quad \boldsymbol{u} \cdot \boldsymbol{n}|_\Gamma = 0\}.$$

Next, we will discuss the properties of $H(curl;\Omega)$. Note that, in the two-dimensional case, the function $\boldsymbol{v} = (v_1, v_2) \in H(curl;\Omega)$ if and only if the function $\boldsymbol{w} = (-v_2, v_1) \in H(div;\Omega)$. Therefore, all the properties of $H(div;\Omega)$ have similar versions for $H(curl;\Omega)$ (cf. [35]).

Theorem 3.1.8. *The space $\mathcal{D}(\bar{\Omega})^2$ is dense in $H(curl;\Omega)$.*

Let us denote the tangential vector of Γ by $\boldsymbol{\tau}$ such that $\boldsymbol{n}$ and $\boldsymbol{\tau}$ obey the right-hand rule. Then we have the following extension theorem ([35, Theorem 2.11]).

Theorem 3.1.9. *The mapping $\gamma_\tau : \boldsymbol{v} \to \boldsymbol{v} \cdot \boldsymbol{\tau}|_\Gamma$ defined on $\mathcal{D}(\bar{\Omega})^2$ can be extended by continuity to a continuous linear mapping, still denoted by γ_τ, from $H(curl;\Omega)$ into $H^{-1/2}(\Gamma)$. Moreover, the following Green's formula holds:*

$$(\boldsymbol{\nabla} \times \boldsymbol{v}, \phi) - (\boldsymbol{v}, \boldsymbol{\nabla} \times \phi) =< \gamma_\tau \boldsymbol{v}, \boldsymbol{\phi} >_\Gamma \quad \forall \boldsymbol{v} \in H(curl;\Omega), \phi \in H^1(\Omega).$$

Remark 3.1.10. We call $\gamma_\tau \boldsymbol{v}$ the tangential trace of $\boldsymbol{v}$ on Γ and it is denoted by $\boldsymbol{v} \cdot \boldsymbol{\tau}$. We also denote $(\gamma_\tau \boldsymbol{v})(\phi)$,for any $\phi \in H^{1/2}(\Gamma)$ and $\gamma_\tau \boldsymbol{v} \in H^{-1/2}(\Gamma)$, by $< \boldsymbol{v} \cdot \boldsymbol{\tau}, \phi >_\Gamma$.

Definition 3.1.11. Let $H_0(curl;\Omega)$ denote

$$Ker(\gamma_\tau) = \{\boldsymbol{u} \in H(curl;\Omega) : \quad \boldsymbol{u} \cdot \boldsymbol{\tau}|_\Gamma = 0\}.$$

The following lemma ([35, Lemma 2.4]) gives us a criterion for $H_0(curl;\Omega)$.

Lemma 3.1.12. *A vector function $\boldsymbol{f}$ of $H(curl;\Omega)$ belongs to $H_0(curl;\Omega)$ if and only if*

$$(\boldsymbol{f}, \boldsymbol{\nabla}\times\phi) - (\boldsymbol{\nabla}\times\boldsymbol{f}, \phi) = 0 \quad \forall\phi\in H^1(\Omega).$$

3.2 Two-dimensional Helmholtz/Hodge Decompositions

In this section we extend the following classical Stokes' Theorem.

Theorem 3.2.1. *If a $\mathcal{C}^1$ vector field has a vanishing curl in a simply-connected region of $\mathbb{R}^2$, then this vector field is the gradient of a function.*

If a $\mathcal{C}^1$ vector field has a vanishing divergence in a simply-connected region of $\mathbb{R}^2$, then this vector field is the curl of a function.

Let us first state a characterization for two-dimensional divergence free vector fields.

3.2.1 Characterization of Two-dimensional Divergence Free Vector Fields and Curl Free Vector Fields

We will not restrict ourselves to a simply-connected domain here. Instead, the domain can be multiply-connected.

Denote by Γ_0 the exterior boundary of Ω and by $\Gamma_i, 1\le i\le p$, the other components of the boundary Γ. Then we have the following characterization for two-dimensional divergence free vector fields.

Theorem 3.2.2. (cf. [35, Theorem 3.1.]) *A vector field $\boldsymbol{v}\in[L_2(\Omega)]^2$ satisfies*

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \quad \textit{and} \quad <\boldsymbol{v}\cdot\boldsymbol{n}, 1>_{\Gamma_i} = 0 \quad \textit{for} \quad 0\le i\le p$$

if and only if there exists a stream function $\phi\in H^1(\Omega)$ such that:

$$\boldsymbol{v} = \boldsymbol{\nabla}\times\phi,$$

where $\boldsymbol{\nabla} \times \phi = (\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1})$. *Moreover* ϕ *is unique up to a constant in* $H^1(\Omega)$.

Remark 3.2.3. If the domain Ω is simply-connected, then $\boldsymbol{v} = \nabla \times \phi$ if and only if $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$.

Similarly, we have the following characterization for two-dimensional curl-free vector fields.

Theorem 3.2.4. *A vector field* $\boldsymbol{v} \in [L_2(\Omega)]^2$ *satisfies:*

$$\boldsymbol{\nabla} \times \boldsymbol{v} = 0 \quad \textit{and} \quad < \boldsymbol{v} \cdot \boldsymbol{\tau}, 1 >_{\Gamma_i} = 0 \quad \textit{for} \quad 0 \le i \le p$$

if and only if there exists a potential function $\phi \in H^1(\Omega)$ *such that*

$$\boldsymbol{v} = \boldsymbol{\nabla} \phi.$$

Moreover ϕ *is unique up to a constant in* $H^1(\Omega)$.

3.2.2 Helmholtz/Hodge Decompositions

There are many different ways to decompose a vector field into a divergence free field and a curl free field. Here, we introduce several Helmholtz/Hodge decompositions for $[L_2(\Omega)]^2$, $H_0(curl;\Omega)$, $H(div^0;\Omega)$ and $H(div^0;\Omega) \cap H_0(curl;\Omega)$, respectively.

We first introduce an orthogonal decomposition for the space $[L_2(\Omega)]^2$ with respect to weighted inner product that will be used in Chapter 4. As in Subsection 3.2.1, the domain Ω is multiply-connected. First we introduce a weighted inner product on the space $[L_2(\Omega)]^2$.

Let ϵ be a bounded positive function in Ω. Define the weighted $L_2(\Omega;\epsilon)$ inner product $(\cdot,\cdot)_{L_2(\Omega;\epsilon)}$ on $[L_2(\Omega)]^2$ by

$$(\boldsymbol{v}, \boldsymbol{w})_{L_2(\Omega;\epsilon)} = \int_\Omega \epsilon(\boldsymbol{v} \cdot \boldsymbol{w}) dx \quad \forall \boldsymbol{v}, \boldsymbol{w} \in [L_2(\Omega)]^2.$$

For distinction we use the notation $[L_2(\Omega;\epsilon)]^2$ for the space $[L_2(\Omega)]^2$ equipped with the weighted inner product $(\cdot,\cdot)_{L_2(\Omega;\epsilon)}$.

Definition 3.2.5. The subspace $H(div^0;\Omega;\epsilon)$ is defined by

$$H(div^0;\Omega;\epsilon) = \{\boldsymbol{v} \in [L_2(\Omega)]^2 : \nabla \cdot (\epsilon \boldsymbol{v}) = 0\}.$$

Definition 3.2.6. The space $\mathcal{H}(\Omega;\epsilon)$ is defined by

$$\begin{aligned}\mathcal{H}(\Omega;\epsilon) = \{\phi \in H^1(\Omega) :& (\epsilon \boldsymbol{\nabla}\phi, \boldsymbol{\nabla} v) = 0 \quad \forall v \in H_0^1(\Omega);\\ & \phi|_{\Gamma_0} = 0; \quad \phi|_{\Gamma_i} = \text{a constant} \quad \text{for} \quad 1 \le i \le p\}.\end{aligned}$$

We have the following decomposition theorem for $[L_2(\Omega;\epsilon)]^2$.

Theorem 3.2.7. *With respect to the weighted $L_2(\Omega;\epsilon)$ inner product, we have the decomposition:*

$$[L_2(\Omega;\epsilon)]^2 = K \oplus H \oplus G, \tag{3.2.1}$$

where

$$K = \epsilon^{-1}\boldsymbol{\nabla} \times H^1(\Omega) = \{\epsilon^{-1}\boldsymbol{\nabla} \times \phi : \phi \in H^1(\Omega)\},$$

$$G = \boldsymbol{\nabla} H_0^1(\Omega) = \{\boldsymbol{\nabla}\phi : \phi \in H_0^1(\Omega)\},$$

and

$$H = \boldsymbol{\nabla}\mathcal{H}(\Omega;\epsilon) = \{\boldsymbol{\nabla}\phi : \phi \in \mathcal{H}(\Omega;\epsilon)\}.$$

Moreover, with respect to the same weighted inner product, we have the decomposition:

$$H(div^0;\Omega;\epsilon) = K \oplus H. \tag{3.2.2}$$

To prepare for the proof of Theorem 3.2.7, we first prove a few lemmas.

Lemma 3.2.8. *We have an orthogonal decomposition*

$$[L_2(\Omega;\epsilon)]^2 = H(div^0;\Omega;\epsilon) \oplus G$$

with respect to the weighted $L_2(\Omega;\epsilon)$ inner product.

Proof. Let $\boldsymbol{v} \in [L_2(\Omega)]^2$. Then

$$\boldsymbol{v} \in H(div^0; \Omega; \epsilon)$$

iff

$$(\epsilon \boldsymbol{v}, \nabla \phi) = 0 \quad \forall \phi \in \mathcal{D}(\Omega)$$

iff

$$(\epsilon \boldsymbol{v}, \nabla \phi) = 0 \quad \forall \phi \in H_0^1(\Omega).$$

In other words, $H(div^0; \Omega; \epsilon)$ is the orthogonal complement of $G = \boldsymbol{\nabla} H_0^1(\Omega)$ with respect to the weighted $L_2(\Omega; \epsilon)$ inner product.

Next we will show that $G = \boldsymbol{\nabla} H_0^1(\Omega)$ is closed in $[L_2(\Omega; \epsilon)]^2$. First we note that it is equivalent to show $G = \boldsymbol{\nabla} H_0^1(\Omega)$ is closed in $[L_2(\Omega)]^2$, since the norms induced by the $L_2(\Omega)$ inner product and the weighted $L_2(\Omega; \epsilon)$ inner product are equivalent on the space $[L_2(\Omega)]^2$.

Let ϕ_n be a sequence in $H_0^1(\Omega)$ such that the sequence $\boldsymbol{\nabla} \phi_n$ converges to a function $\boldsymbol{v}$ in $[L_2(\Omega)]^2$ and hence $\boldsymbol{\nabla} \phi_n$ is a Cauchy sequence in $[L_2(\Omega)]^2$. Because of Poincare's inequality (cf. Proposition (5.3.5), [20]), ϕ_n is a Cauchy sequence in $H_0^1(\mathbf{O})$ and hence the sequence ϕ_n converges to a function ϕ in $H_0^1(\Omega)$ and hence $\boldsymbol{\nabla} \phi_n$ converges to a function $\boldsymbol{\nabla} \phi$ in $[L_2(\Omega)]^2$. So $\boldsymbol{v} = \boldsymbol{\nabla} \phi$, where $\phi \in H_0^1(\Omega)$. Therefore $G = \boldsymbol{\nabla} H_0^1(\Omega)$ is closed in $[L_2(\Omega)]^2$.

Since $G = \boldsymbol{\nabla} H_0^1(\Omega)$ is closed in $[L_2(\Omega; \epsilon)]^2$ and $H(div^0; \Omega; \Omega)$ is the orthogonal complement of $G = \boldsymbol{\nabla} H_0^1(\Omega)$ in $[L_2(\Omega; \epsilon)]^2$, we have the orthogonal decomposition

$$[L_2(\Omega; \epsilon)]^2 = H(div^0; \Omega; \epsilon) \oplus G$$

with respect to the weighted $L_2(\Omega; \epsilon)$ inner product. $\square$

Lemma 3.2.9. *Let $\zeta \in H^1(\Omega)$ such that the trace of ζ on Γ_i is a constant γ_i for $0 \le i \le p$. Then we have*

$$(\boldsymbol{\nabla} \times \psi, \boldsymbol{\nabla}\zeta) = 0 \quad \forall \psi \in H^1(\Omega). \tag{3.2.3}$$

Proof. Let $\boldsymbol{v} = \boldsymbol{\nabla} \times \psi$. By Theorem 3.2.2, we have

$$< \boldsymbol{v} \cdot \boldsymbol{n}, 1 >_{\Gamma_i} = 0 \quad \forall 0 \le i \le p. \tag{3.2.4}$$

So Corollary 3.1.5 and (3.2.4) imply that

$$\begin{aligned}(\boldsymbol{\nabla} \times \psi, \boldsymbol{\nabla}\zeta) = (\boldsymbol{v}, \boldsymbol{\nabla}\zeta) &= \sum_{i=0}^{p} < \boldsymbol{v} \cdot \boldsymbol{n}, \zeta >_{\Gamma_i} \\ &= \sum_{i=0}^{p} \zeta|_{\Gamma_i} < \boldsymbol{v} \cdot \boldsymbol{n}, 1 >_{\Gamma_i} = 0.\end{aligned}$$

□

This lemma leads to the following corollary.

Corollary 3.2.10. *We have $\nabla H_0^1(\Omega) \subset H_0(curl; \Omega)$.*

Proof. Obviously, $\boldsymbol{\nabla} H_0^1(\Omega) \subset H(curl; \Omega)$. Lemma 3.2.9 implies that for any $\zeta \in H_0^1(\Omega)$,

$$(\boldsymbol{\nabla} \times \psi, \boldsymbol{\nabla}\zeta) = 0 = (\psi, \boldsymbol{\nabla} \times \boldsymbol{\nabla}\zeta) \quad \forall \psi \in H^1(\Omega). \tag{3.2.5}$$

So $\boldsymbol{\nabla}\zeta \in H_0(curl; \Omega)$ by Lemma 3.1.12 and hence $\nabla H_0^1(\Omega) \subset H_0(curl; \Omega)$. □

Lemma 3.2.11. *Let $\varphi \in \mathcal{H}(\Omega; \epsilon)$. Then*

$$\int_{\Gamma_i} \epsilon \frac{\partial \varphi}{\partial n} ds = 0 \quad \textit{for} \quad 1 \le i \le p \tag{3.2.6}$$

if and only if $\varphi = 0$.

Proof. It is sufficient to show that if $\varphi \in \mathcal{H}(\Omega;\epsilon)$ and φ satisfies (3.2.6), then $\varphi = 0$. By Corollary 3.1.5 and Definition 3.2.6, we have

$$\begin{aligned}(\epsilon\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}\varphi) &= \sum_{i=0}^{p} < \epsilon\frac{\partial\varphi}{\partial n}, \varphi >_{\Gamma_i} \\ &= \sum_{i=0}^{p} \varphi|_{\Gamma_i} < \epsilon\frac{\partial\varphi}{\partial n}, 1 >_{\Gamma_i} = 0.\end{aligned}$$

So $\varphi = 0$, since the domain Ω is connected and $\varphi = 0$ on Γ_0. □

Lemma 3.2.12. *We have an orthogonal decomposition*

$$H(div^0;\Omega;\epsilon) = K \oplus H$$

with respect to the weighted $L_2(\Omega;\epsilon)$ inner product.

Proof. Lemma 3.2.9 implies that K and H are orthogonal to each other under the weighted $L_2(\Omega;\epsilon)$ inner product. H is also closed, since it is a finite dimensional space. Next we will show that K is closed in $[L_2(\Omega;\epsilon)]^2$.

First we note that it is equivalent to show K is closed in $[L_2(\Omega)]^2$, since the norms induced by the $L_2(\Omega)$ inner product and the weighted $L_2(\Omega;\epsilon)$ inner product are equivalent on the space $[L_2(\Omega)]^2$. Furthermore, it is equivalent to show that $\boldsymbol{\nabla} \times H^1(\Omega)$ is closed $[L_2(\Omega)]^2$.

Now let ϕ_n be a sequence in $H^1(\Omega)$ such that the sequence $\boldsymbol{\nabla}\times\phi_n$ converges to a function $\boldsymbol{v}$ in $[L_2(\Omega)]^2$ and hence $\boldsymbol{\nabla}\times\phi_n$ is a Cauchy sequence in $[L_2(\Omega)]^2$. Because of Friedrichs' inequality (cf. [20, Lemma 4.3.14]), it follows that the sequence $\phi_n - \bar{\phi}_n$ is a Cauchy sequence in $H^1(\Omega)$ and hence converges to a function ϕ in $H^1(\Omega)$, where $\bar{\phi}_n = \frac{1}{|\Omega|}\int_\Omega \phi_n dx$. Hence $\boldsymbol{\nabla}\times\phi_n = \boldsymbol{\nabla}\times(\phi_n - \bar{\phi}_n)$ converges to the function $\boldsymbol{\nabla}\times\phi$ in $[L_2(\Omega)]^2$. So $\boldsymbol{v} = \boldsymbol{\nabla}\times\phi$, where $\phi \in H^1(\Omega)$. Therefore $\boldsymbol{\nabla}\times H^1(\Omega)$ is closed in $[L_2(\Omega)]^2$.

The remaining task of the proof is to show that $H(div^0;\Omega;\epsilon) = K + H$.

Let $\boldsymbol{v}$ be a vector function in $H(div^0; \Omega; \epsilon)$, there exists a function $\varphi \in H$ such that

$$< \epsilon \frac{\partial \varphi}{\partial n}, 1 >_{\Gamma_i} = < \epsilon \boldsymbol{v} \cdot \boldsymbol{n}, 1 >_{\Gamma_i} \quad \forall 1 \le i \le p,$$

since by Lemma 3.2.11, the mapping defined by

$$\varphi \rightarrow \begin{bmatrix} < \epsilon \boldsymbol{\nabla} \varphi \cdot \boldsymbol{n}, 1 >_{\Gamma_1} \\ < \epsilon \boldsymbol{\nabla} \varphi \cdot \boldsymbol{n}, 1 >_{\Gamma_2} \\ \vdots \\ < \epsilon \boldsymbol{\nabla} \varphi \cdot \boldsymbol{n}, 1 >_{\Gamma_p} \end{bmatrix}$$

is an isomorphism from $\mathcal{H}(\Omega; \epsilon)$ to $\mathbb{R}^p$.

Now $\boldsymbol{\nabla} \cdot (\epsilon \boldsymbol{v} - \epsilon \boldsymbol{\nabla} \varphi) = 0$ and

$$< (\epsilon \boldsymbol{v} - \epsilon \boldsymbol{\nabla} \varphi) \cdot \boldsymbol{n}, 1 >_{\Gamma_i} = 0 \quad \text{for} \quad 1 \le i \le p.$$

Therefore, by Theorem 3.2.2, there exists $\phi \in H^1(\Omega)$ such that $\epsilon \boldsymbol{v} - \epsilon \boldsymbol{\nabla} \varphi = \boldsymbol{\nabla} \times \phi$. Let $\boldsymbol{v}_1 = \epsilon^{-1} \boldsymbol{\nabla} \times \phi$, then we have

$$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{\nabla} \varphi,$$

where $\boldsymbol{v}_1 \in K$ and $\varphi \in HG$. $\square$

Proof. (Proof of Theorem 3.2.7) Using Lemma 3.2.8 and Lemma 3.2.12, we have Theorem 3.2.7. $\square$

Theorem 3.2.7 leads to the decomposition for $H_0(curl; \Omega)$ and $H_0(curl; \Omega) \cap H(div; \Omega; \epsilon)$ in the following corollaries.

Corollary 3.2.13. (Decomposition for $H_0(curl; \Omega)$) *Suppose that $\boldsymbol{v} \in H_0(curl; \Omega)$, then there exist a unique $\mathring{\boldsymbol{v}}$ and φ such that*

$$\boldsymbol{v} = \mathring{\boldsymbol{v}} + \boldsymbol{\nabla} \varphi,$$

where $\mathring{\boldsymbol{v}} \in H_0(curl; \Omega) \cap H(div^0; \Omega; \epsilon)$ and $\varphi \in H_0^1(\Omega)$.

Proof. Since $\boldsymbol{v} \in H_0(curl;\Omega) \subset [L_2(\Omega)]^2$, by Theorem 3.2.7, there exists a unique $\mathring{\boldsymbol{v}} \in H(div^0;\Omega;\epsilon)$ and $\varphi \in H_0^1(\Omega)$ such that

$$\boldsymbol{v} = \mathring{\boldsymbol{v}} + \boldsymbol{\nabla}\varphi.$$

Note that $\boldsymbol{\nabla}\varphi \in H_0(curl;\Omega)$ by Lemma 3.2.9 and $\boldsymbol{v} \in H_0(curl;\Omega)$, which imply that $\mathring{\boldsymbol{v}} \in H_0(curl;\Omega)$. □

A similar argument as in the proof of Corollary 3.2.13 leads to the following decomposition.

Corollary 3.2.14. (Decomposition for $H_0(curl;\Omega) \cap H(div;\Omega;\epsilon)$) *Suppose that $\boldsymbol{v} \in H_0(curl;\Omega) \cap H(div;\Omega;\epsilon)$, then there exist a unique $\mathring{\boldsymbol{v}}$ and φ such that*

$$\boldsymbol{v} = \mathring{\boldsymbol{v}} + \boldsymbol{\nabla}\varphi,$$

where $\mathring{\boldsymbol{v}} \in H_0(curl;\Omega) \cap H(div^0;\Omega;\epsilon)$ and $\varphi \in H_0^1(\Omega)$. Moreover φ satisfies

$$(\epsilon\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}\phi) = (\boldsymbol{\nabla}\cdot(\epsilon\boldsymbol{v}), \phi) \quad \forall \phi \in H_0^1(\Omega).$$

Chapter 4
Maxwell Equations in Homogeneous Media

4.1 Introduction

For simplicity, we assume that $\epsilon = 1$ and $\mu = 1$. We will follow the notation introduced in Subsection 3.2.2 but with ϵ suppressed. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, Γ_0 the exterior boundary, $\Gamma_1, \cdots \Gamma_p$ the components of the interior boundary, and α be a constant. Then the weak formulation of the Maxwell's equations is:

For $\boldsymbol{f} \in [L_2(\Omega)]^2$, find $\mathring{\boldsymbol{u}} \in H_0(curl;\Omega) \cap H(div^0;\Omega)$ such that

$$(\boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}, \boldsymbol{\nabla} \times \boldsymbol{v}) + \alpha(\mathring{\boldsymbol{u}}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in H_0(curl;\Omega) \cap H(div^0;\Omega). \tag{4.1.1}$$

We assume that $-\alpha$ is not a Maxwell eigenvalue so that (4.1.1) is uniquely solvable.

Remark 4.1.1. For $\alpha \leq 0$, (4.1.1) is exactly the weak form for (1.1.38). For $\alpha > 0$, (4.1.1) is related to the time-domain Maxwell's equations.

We will use the Helmholtz/Hodge decomposition from Chapter 3 to reduce (4.1.1) to standard second order scalar elliptic boundary value problems. The presentation in this chapter follows [14] closely.

4.2 Equation for $\xi = \boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}$ and ϕ

Because of the Helmholtz/Hodge decomposition for $H(div^0;\Omega)$ (See Theorem 3.2.7), we can reformulate the problem (4.1.1) as coupled elliptic problems.

Theorem 4.2.1. *Suppose that the solution $\mathring{\boldsymbol{u}} \in H_0(curl;\Omega) \cap H(div^0;\Omega)$ is decomposed as in Theorem 3.2.7,*

$$\mathring{\boldsymbol{u}} = \boldsymbol{\nabla} \times \phi + \boldsymbol{\nabla}\varphi, \tag{4.2.1}$$

where $\varphi \in \mathcal{H}(\Omega)$ *and* $\phi \in H^1(\Omega)$ *satisfies* $(\phi, 1) = 0$*. Then* ϕ *is determined by*

$$(\nabla \times \phi, \nabla \times \psi) = (\xi, \psi) \quad \forall \psi \in H^1(\Omega) \tag{4.2.2}$$

and the constraint

$$(\phi, 1) = 0, \tag{4.2.3}$$

where the function $\xi = \nabla \times \mathring{\boldsymbol{u}} \in H^1(\Omega)$ *is determined by*

$$(\nabla \times \xi, \nabla \times \psi) + \alpha(\xi, \psi) = (\boldsymbol{f}, \nabla \times \psi) \quad \forall \psi \in H^1(\Omega) \tag{4.2.4}$$

when $\alpha \neq 0$*, and by (4.2.4) together with the constraint*

$$(\xi, 1) = 0 \tag{4.2.5}$$

when Ω *is simply connected and* $\alpha = 0$*. Moreover, when* $p \geq 1$ *and* $\alpha \neq 0$*,* φ *can be determined by*

$$(\nabla\varphi, \nabla\psi) = \frac{1}{\alpha}(\boldsymbol{f}, \nabla\psi) \quad \forall \psi \in \mathcal{H}(\Omega). \tag{4.2.6}$$

To prove Theorem 4.2.1, we need the following lemma concerning with the strong form of (4.1.1).

Lemma 4.2.2. *The solution* $\mathring{\boldsymbol{u}}$ *of (4.1.1) satisfies*

$$\nabla \times (\nabla \times \mathring{\boldsymbol{u}}) + \alpha \mathring{\boldsymbol{u}} = Q\boldsymbol{f}$$

in the sense of distribution, where $Q : [L_2(\Omega)]^2 \to H(div^0; \Omega)$ *is the orthogonal projection.*

Proof. Let $\boldsymbol{\zeta} \in [\mathcal{D}(\Omega)]^2$ be a C^∞ vector field with compact support in Ω. So $\boldsymbol{\zeta} \in H_0(curl; \Omega)$, $Q\boldsymbol{\zeta} \in H(div^0; \Omega)$ and $\boldsymbol{\zeta} - Q\boldsymbol{\zeta} \in \nabla H_0^1(\Omega)$ by Lemma 3.2.8 .

Because of Corollary 3.2.10, we have $\boldsymbol{\zeta} - Q\boldsymbol{\zeta} \in H_0(curl; \Omega)$ and hence $Q\boldsymbol{\zeta} \in H_0(curl; \Omega)$. Therefore,

$$Q\boldsymbol{\zeta} \in H_0(curl; \Omega) \cap H(div^0; \Omega). \tag{4.2.7}$$

Furthermore, we have

$$\nabla \times (\boldsymbol{\zeta} - Q\boldsymbol{\zeta}) = 0, \tag{4.2.8}$$

since $\nabla \times (\nabla H_0^1(\Omega)) = \{0\}$, and for $\mathring{\boldsymbol{u}} \in H(div^0; \Omega)$, we have

$$(\mathring{\boldsymbol{u}}, \boldsymbol{\zeta} - Q\boldsymbol{\zeta}) = 0, \tag{4.2.9}$$

by Lemma 3.2.8 and the fact that $\boldsymbol{\zeta} - Q\boldsymbol{\zeta} \in \nabla H_0^1(\Omega)$.

Using (4.1.1), (4.2.7), (4.2.8) and (4.2.9), we have

$$\begin{aligned}(\nabla \times \mathring{\boldsymbol{u}}, \nabla \times \boldsymbol{\zeta}) + \alpha(\mathring{\boldsymbol{u}}, \boldsymbol{\zeta}) =& (\nabla \times \mathring{\boldsymbol{u}}, \nabla \times (Q\boldsymbol{\zeta} + (\boldsymbol{\zeta} - Q\boldsymbol{\zeta}))) \\ & + \alpha(\mathring{\boldsymbol{u}}, Q\boldsymbol{\zeta} + (\boldsymbol{\zeta} - Q\boldsymbol{\zeta})) \\ =& (\nabla \times \mathring{\boldsymbol{u}}, \nabla \times Q\boldsymbol{\zeta}) + \alpha(\mathring{\boldsymbol{u}}, Q\boldsymbol{\zeta}) \\ =& (\boldsymbol{f}, Q\boldsymbol{\zeta}) = (Q\boldsymbol{f}, \boldsymbol{\zeta}),\end{aligned}$$

which completes the proof. □

Remark 4.2.3. Lemma 4.2.2 implies that $\xi = \nabla \times \mathring{\boldsymbol{u}} \in H^1(\Omega)$ and

$$\nabla \times \xi + \alpha\mathring{\boldsymbol{u}} = Q\boldsymbol{f}. \tag{4.2.10}$$

Now we are ready to prove Theorem 4.2.1.

Proof. (Proof of Theorem 4.2.1) First, let us justify (4.2.2) by using (4.2.1), Lemma 3.1.12 and Lemma 3.2.9. Let $\psi \in H^1(\Omega)$ be arbitrary. We have

$$\begin{aligned}(\nabla \times \phi, \nabla \times \psi) &= (\nabla \times \phi + \nabla\varphi, \nabla \times \psi) = (\mathring{\boldsymbol{u}}, \nabla \times \varphi) \\ &= (\nabla \times \mathring{\boldsymbol{u}}, \varphi) = (\xi, \varphi).\end{aligned}$$

To justify (4.2.6) when $p \geq 1$, we take $\boldsymbol{v} = \nabla\psi$ in (4.1.1) where $\psi \in \mathcal{H}(\Omega)$, and replace $\mathring{\boldsymbol{u}}$ by the Hodge/Helmholtz decomposition (4.2.1). Then we obtain the equation

$$\alpha(\nabla \times \phi + \nabla\varphi, \nabla\psi) = (\boldsymbol{f}, \nabla\psi),$$

which implies (4.2.6) by Lemma 3.2.9.

Now let us justify (4.2.4). Let $\psi \in H^1(\Omega)$ be arbitrary. Since $\boldsymbol{\nabla} \times \psi \in H(div^0; \Omega)$ by Lemma 3.2.2, we have

$$
\begin{aligned}
(\boldsymbol{f}, \boldsymbol{\nabla} \times \psi) &= (Q\boldsymbol{f}, \boldsymbol{\nabla} \times \psi) \\
&= (\boldsymbol{\nabla} \times \xi + \alpha \mathring{\boldsymbol{u}}, \boldsymbol{\nabla} \times \psi) \quad \text{(by the equation (4.2.10))} \\
&= (\boldsymbol{\nabla} \times \xi, \boldsymbol{\nabla} \times \psi) + \alpha(\boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}, \psi) \quad \text{(by Lemma 3.1.12)} \\
&= (\boldsymbol{\nabla} \times \xi, \boldsymbol{\nabla} \times \psi) + \alpha(\xi, \psi),
\end{aligned}
$$

which gives (4.2.4). The constraint $(\xi, 1) = 0$ follows immediately from Lemma 3.1.12.

$\square$

Note that $\mathcal{H}(\Omega)$ is a finite dimensional space with the basis $\{\varphi_1, \varphi_2, \cdots, \varphi_p\}$, where $\varphi_i, 1 \leq i \leq p$ satisfies that

$$\Delta \varphi_i = 0, \tag{4.2.11a}$$

$$\varphi_i|_{\Gamma_0} = 0, \tag{4.2.11b}$$

$$\varphi_i|_{\Gamma_j} = \delta_{ij} \quad \text{for } 1 \leq j \leq p, \tag{4.2.11c}$$

i.e.,

$$(\boldsymbol{\nabla} \varphi_i, \boldsymbol{\nabla} v) = 0 \quad \text{for} \quad v \in H_0^1(\Omega), \tag{4.2.12a}$$

$$\varphi_i|_{\Gamma_0} = 0, \tag{4.2.12b}$$

$$\varphi_i|_{\Gamma_j} = \delta_{ij} \quad \text{for} \quad 1 \leq j \leq p, \tag{4.2.12c}$$

therefore φ in (4.2.6) can be written as

$$\sum_{i=1}^{p} c_i \varphi_i, \tag{4.2.13}$$

where the coefficients c_i's are determined by the symmetric positive-definite system

$$\sum_{i=1}^{p}(\boldsymbol{\nabla}\varphi_i, \boldsymbol{\nabla}\varphi_k)c_i = \frac{1}{\alpha}(\boldsymbol{f}, \boldsymbol{\nabla}\varphi_k) \quad \text{for } 1 \leq k \leq p. \tag{4.2.14}$$

Next we discuss the relation between the solvability of (4.1.1) and the solvability of (4.2.4) under the condition that $-\alpha(\neq 0)$ is not a Maxwell eigenvalue.

Lemma 4.2.4. *For $\alpha \neq 0$, the problem (4.1.1) is uniquely solvable if and only if the problem (4.2.4) is uniquely solvable.*

Proof. Let α be nonzero. Since $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$ by the Rellich-Kondrachov theorem [1] and $H_0(curl;\Omega) \cap H(div^0;\Omega)$ is compactly embedded in $[L_2(\Omega)]^2$ by a result of Weber [62], we can apply the Fredholm alternative [33] to consider only the homogeneous equation corresponding to (4.1.1)

$$(\boldsymbol{\nabla} \times \boldsymbol{w}, \boldsymbol{\nabla} \times \boldsymbol{v}) + \alpha(\boldsymbol{w}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in H_0(curl;\Omega) \cap H(div^0;\Omega), \tag{4.2.15}$$

and the homogeneous equation corresponding to (4.2.4)

$$(\boldsymbol{\nabla} \times \eta, \boldsymbol{\nabla} \times \psi) + \alpha(\eta, \psi) = 0 \quad \forall \psi \in H^1(\Omega). \tag{4.2.16}$$

By the Fredholm alternative (cf. [34, Theorem 5.11]), it suffices to show that (4.2.15) has a nontrivial solution $\boldsymbol{w} \in H_0(curl;\Omega) \cap H(div^0;\Omega)$ if and only if (4.2.16) has a nontrivial solution $\eta \in H^1(\Omega)$.

Suppose there exists a nontrivial $\boldsymbol{w} \in H_0(curl;\Omega) \cap H(div^0;\Omega)$ that satisfies (4.2.15). Let $\eta = \boldsymbol{\nabla} \times \boldsymbol{w}$, then $\eta \in H^1(\Omega)$ and (4.2.16) holds as a special case of (4.2.4) where $\boldsymbol{f} = 0$.

Suppose there exists a nontrivial $\eta \in H^1(\Omega)$ that satisfies (4.2.16). Since $\alpha \neq 0$, we deduce from (4.2.16) that $(\eta, 1) = 0$. Let $\boldsymbol{w} = \boldsymbol{\nabla} \times \rho$, where $\rho \in H^1(\Omega)$ is defined by the Neumann problem

$$(\boldsymbol{\nabla} \times \rho, \boldsymbol{\nabla} \times \psi) = (\eta, \psi) \quad \psi \in H^1(\Omega), \tag{4.2.17a}$$

$$(\rho, 1) = 0. \tag{4.2.17b}$$

Then Theorem 3.2.2 and (4.2.17a) imply

$$\eta = \boldsymbol{\nabla} \times \boldsymbol{w} \tag{4.2.18}$$

and $\boldsymbol{w} \in H(div^0; \Omega)$. Note that $\boldsymbol{w} \in H(curl; \Omega)$ by (4.2.16). Since (4.2.17a) can also be written as

$$(\boldsymbol{w}, \boldsymbol{\nabla} \times \psi) = (\boldsymbol{\nabla} \times \boldsymbol{w}, \psi) \quad \forall \psi \in H^1(\Omega),$$

we have $\boldsymbol{w} \in H_0(curl; \Omega)$ by Lemma 3.1.12. Therefore, $\boldsymbol{w} \in H_0(curl; \Omega) \cap H(div^0; \Omega)$.

It is easy to see that $\boldsymbol{w}$ is nontrivial. To check that it satisfies (4.2.15), we take an arbitrary $\boldsymbol{v} \in H_0(curl; \Omega) \cap H(div^0; \Omega)$ and write its Hodge decomposition (cf. Theorem 3.2.7) as

$$\boldsymbol{v} = \boldsymbol{\nabla} \times \phi + \boldsymbol{\nabla} \varphi, \tag{4.2.19}$$

where $\phi \in H^1(\Omega)$ and $\varphi \in \mathcal{H}(\Omega)$. Note that, by Lemma 3.2.9, we have

$$(\boldsymbol{\nabla} \times \eta, \boldsymbol{\nabla} \varphi) = 0 \quad \text{and} \quad (\boldsymbol{w}, \boldsymbol{\nabla} \varphi) = (\boldsymbol{\nabla} \times \rho, \boldsymbol{\nabla} \varphi) = 0. \tag{4.2.20}$$

It follows from Lemma 3.1.12, (4.2.16), (4.2.18), (4.2.19) and (4.2.20) that

$$\begin{aligned}
(\boldsymbol{\nabla} \times \boldsymbol{w}, \boldsymbol{\nabla} \times \boldsymbol{v}) = (\eta, \boldsymbol{\nabla} \times \boldsymbol{v}) &= (\boldsymbol{\nabla} \times \eta, \boldsymbol{v}) \\
&= (\boldsymbol{\nabla} \times \eta, \boldsymbol{\nabla} \times \phi + \boldsymbol{\nabla} \varphi) \\
&= (\boldsymbol{\nabla} \times \eta, \boldsymbol{\nabla} \times \phi) \\
&= -\alpha(\eta, \phi) \\
&= -\alpha(\boldsymbol{\nabla} \times \boldsymbol{w}, \phi) \\
&= -\alpha(\boldsymbol{w}, \boldsymbol{\nabla} \times \phi) \\
&= -\alpha(\boldsymbol{w}, \boldsymbol{\nabla} \times \phi + \boldsymbol{\nabla} \varphi) = -\alpha(\boldsymbol{w}, v),
\end{aligned}$$

i.e., $\boldsymbol{w}$ satisfies (4.2.15).

$\square$

It follows from Theorem 4.2.1 and Lemma 4.2.4 that we can solve (4.1.1) by the following numerical procedure under the assumption that $-\alpha$ is not a Maxwell eigenvalue.

Step 1. Compute a numerical approximation $\tilde{\xi}$ of ξ by solving (4.2.4) when $\alpha \neq 0$, and by solving (4.2.4) with the constraint $(\xi, 1) = 0$ when Ω is simply connected and $\alpha = 0$.

Step 2. Compute a numerical approximation $\tilde{\phi}$ of ϕ by solving (4.2.2) under the constraint $(\phi, 1) = 0$, where ξ is replaced by $\tilde{\xi}$.

Step 3. Compute numerical approximations $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_p$ of $\varphi_1, \cdots, \varphi_p$ by solving the boundary value problems in (4.2.11).

Step 4. Compute numerical approximations $\tilde{c}_1, \cdots, \tilde{c}_p$ by solving (4.2.14), where $\varphi_1, \cdots, \varphi_p$ are replace by $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_p$.

Step 5. The numerical approximation $\tilde{\boldsymbol{u}}$ for $\mathring{\boldsymbol{u}}$ is given by

$$\tilde{\boldsymbol{u}} = \boldsymbol{\nabla} \times \tilde{\phi} + \sum_{i=1}^{p} \tilde{c}_i \boldsymbol{\nabla} \tilde{\varphi}_i.$$

4.3 A P_1 Finite Element Method

In this section, we use a P_1 finite element method to demonstrate our approach.

Let $\mathscr{T}_h$ be a quasi-uniform simplicial triangulation of Ω with mesh size h and $V_h \subset H^1(\Omega)$ be the P_1 finite element space associated with $\mathscr{T}_h$ (See Section 2.4).

For $\alpha \neq 0$, the P_1 finite element method for (4.2.4) is to find $\xi_h \in V_h$ such that

$$(\boldsymbol{\nabla} \times \xi_h, \boldsymbol{\nabla} \times v) + \alpha(\xi_h, v) = (\boldsymbol{f}, \boldsymbol{\nabla} \times v) \quad \forall v \in V_h. \tag{4.3.1}$$

For $\alpha > 0$, the problem (4.3.1) is symmetric positive-definite and hence well-posed. It is also well-posed for $\alpha < 0$ provided $-\alpha$ is not a Maxwell eigenvalue and h is sufficiently small (cf. Lemma 4.4.2).

Note that when $\alpha \neq 0$ (4.3.1) implies

$$(\xi_h, 1) = 0. \tag{4.3.2}$$

When Ω is simply connected and $\alpha = 0$, $\xi_h \in V_h$ is determined by (4.3.1) together with the constraint (4.3.2). It is a well-posed problem because of the Poincare-Friedrichs inequality (cf. [20])

$$\|v\|_{L_2(\Omega)} \leq C(|(v,1)| + \|\boldsymbol{\nabla} \times v\|_{L_2(\Omega)}) \quad \forall v \in H^1(\Omega). \tag{4.3.3}$$

The P_1 finite element approximation ϕ_h of ϕ is then determined by

$$(\boldsymbol{\nabla} \times \phi_h, \boldsymbol{\nabla} \times v) = (\xi_h, v) \quad \forall v \in V_h, \tag{4.3.4a}$$

$$(\phi_h, 1) = 0. \tag{4.3.4b}$$

The problem (4.3.4) is well-posed because of (4.3.2) and (4.3.3).

For the multiply connected domain Ω (i.e., $p \neq 0$), we have the approximation φ_h of φ as follows:

$$\varphi_h = \sum_{i=1}^{p} c_{i,h} \boldsymbol{\nabla} \varphi_{i,h},$$

where $\varphi_{i,h}$ is determined by

$$(\boldsymbol{\nabla}\varphi_{i,h}, \boldsymbol{\nabla} v) = 0 \quad \forall v \in \mathring{V}_h = V_h \cap H_0^1(\Omega), \tag{4.3.5a}$$

$$\varphi_{i,h}|_{\Gamma_0} = 0, \tag{4.3.5b}$$

$$\varphi_{i,h}|_{\Gamma_k} = \delta_{jk} \quad \text{for } 1 \leq k \leq p, \tag{4.3.5c}$$

and the $c_{i,h}$'s are determined by

$$\sum_{i=1}^{p} (\boldsymbol{\nabla}\varphi_{i,h}, \boldsymbol{\nabla}\varphi_{k,h}) c_{i,h} = \frac{1}{\alpha}(\boldsymbol{f}, \boldsymbol{\nabla}\varphi_{k,h}) \quad \text{for } 1 \leq k \leq p. \tag{4.3.6}$$

Finally, we approximate $\mathring{\boldsymbol{u}}$ by the piecewise constant vector field $\mathring{\boldsymbol{u}}_h$ defined by

$$\mathring{\boldsymbol{u}}_h = \boldsymbol{\nabla} \times \phi_h + \sum_{i=1}^{p} c_{i,h} \boldsymbol{\nabla} \varphi_{i,h}. \tag{4.3.7}$$

4.4 Convergence Analysis

In this section, we use standard techniques to analyze the P_1 finite element method in Section 4.3, since (4.2.2), (4.2.1) and (4.2.6) only involve standard second order scalar elliptic problems. Before doing this, let us introduce the related interpolation error estimates which are similar to the one introduced in Section 2.4.

Let the index β be defined by

$$\beta = \min(1, \min_{1 \leq l \leq N_\Omega} \frac{\pi}{\omega_l}), \tag{4.4.1}$$

where $\omega_1, \omega_2, \ldots, \omega_{N_\Omega}$ are the interior angles at the corners of Ω.

We have the following estimate for the solution of (4.3.5):

$$\|\varphi_i - \Pi_h \varphi_i\|_{L_2(\Omega)} + h|\varphi_i - \Pi_h \varphi_i|_{H^1(\Omega)} \leq Ch^{1+\beta}, \tag{4.4.2}$$

where Π_h is the nodal interpolation operator for the P_1 finite element.

Similarly, for the solution ζ of the Laplace equation with homogeneous Neumann boundary condition, we have

$$\|\zeta - \Pi_h \zeta\|_{L_2(\Omega)} + h|\zeta - \Pi_h \zeta|_{H^1(\Omega)} \leq Ch^{1+\beta} \|g\|_{L_2(\Omega)}, \tag{4.4.3}$$

where g is the right-hand side function.

We begin by comparing ξ_h and $\xi = \boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}$. The following result is obtained by using (4.2.4), (4.3.1), (4.4.3) and a standard duality argument.

Lemma 4.4.1. *For $\alpha > 0$ (general Ω) and $\alpha = 0$ (simply connected Ω), we have*

$$\|\xi - \xi_h\|_{L_2(\Omega)} \leq Ch^\beta \inf_{v \in V_h} \|\boldsymbol{\nabla} \times (\xi - v)\|_{L_2(\Omega)}. \tag{4.4.4}$$

Proof. Combining (4.2.4) and (4.3.1), we have the Galerkin orthogonality

$$(\boldsymbol{\nabla} \times (\xi - \xi_h), \boldsymbol{\nabla} \times v) + \alpha((\xi - \xi_h), v) = 0 \quad \text{for} \quad v \in V_h. \tag{4.4.5}$$

From (4.4.5) we conclude that

$$((\xi - \xi_h), 1) = 0 \tag{4.4.6}$$

when $\alpha \neq 0$. If $\alpha = 0$ (simply connected Ω), then, from (4.2.5) and (4.3.2), we have the equation (4.4.6).

Since $\xi, \xi_h \in H^1(\Omega)$, it follows from (4.4.6) and Poincaré-Freidrichs inequality (cf. [20, (10.6.1)]) that

$$\|\xi - \xi_h\|_{L_2(\Omega)} \leq C|\xi - \xi_h|_{H^1(\Omega)}, \tag{4.4.7}$$

where the positive constant C depends only on the domain Ω.

Now we estimate $|\xi - \xi_h|_{H^1(\Omega)}$. Let $v \in V_h$. It follows from (4.4.5) and (4.4.7) that

$$\begin{aligned}
|\xi - \xi_h|^2_{H^1(\Omega)} + \alpha\|\xi - \xi_h\|^2_{L_2(\Omega)} &= (\boldsymbol{\nabla} \times (\xi - \xi_h), \boldsymbol{\nabla} \times (\xi - \xi_h)) + \alpha(\xi - \xi_h, \xi - \xi_h) \\
&= (\boldsymbol{\nabla} \times (\xi - \xi_h), \boldsymbol{\nabla} \times (\xi - v)) + \alpha(\xi - \xi_h, \xi - v) \\
&\leq C|\xi - \xi_h|_{H^1(\Omega)}|\xi - v|_{H^1(\Omega)},
\end{aligned}$$

which implies

$$|\xi - \xi_h|_{H^1(\Omega)} \leq C|\xi - v|_{H^1(\Omega)} \quad \forall v \in V_h. \tag{4.4.8}$$

We prove an error estimate for $\xi - \xi_h$ in the L_2 norm by a duality argument.

Let ζ be the solution of

$$(\boldsymbol{\nabla} \times \zeta, \boldsymbol{\nabla} \times v) + \alpha(\zeta, v) = (e, v) \quad \forall v \in H^1(\Omega), \tag{4.4.9}$$

where $e = \xi - \xi_h$. When $\alpha = 0$, (4.4.9) is uniquely solvable up to an additive constant (cf. [20, Section 5.2]) and we assume its solution ζ satisfying the constant

$(\zeta, 1) = 0$. It follows from (4.4.3), (4.4.5), (4.4.7), (4.4.8), and (4.4.9) that

$$
\begin{aligned}
\|\xi - \xi_h\|_{L_2(\Omega)}^2 =& (e, \xi - \xi_h) \\
=& (\nabla \times \zeta, \nabla \times (\xi - \xi_h)) + \alpha(\zeta, \xi - \xi_h) \\
=& (\nabla \times (\zeta - \Pi_h \zeta), \nabla \times (\xi - \xi_h)) + \alpha(\zeta - \Pi_h \zeta, \xi - \xi_h) \\
\leq& C(|\zeta - \Pi_h \zeta|_{H^1(\Omega)} + \|\zeta - \Pi_h \zeta\|_{L_2(\Omega)}) \\
& \times (|\xi - \xi_h|_{H^1(\Omega)} + \|\xi - \xi_h\|_{L_2(\Omega)}) \\
\leq& C h^\beta \|\xi - \xi_h\|_{L_2(\Omega)} |\xi - \xi_h|_{H^1(\Omega)} \qquad (4.4.10) \\
\leq& C h^\beta \|\xi - \xi_h\|_{L_2(\Omega)} \inf_{v \in V_h} |\xi - v|_{H^1(\Omega)},
\end{aligned}
$$

which implies (4.4.4). □

In the case $\alpha < 0$, we have the following result by using the approach of Schatz [58], where the required well-posedness of the continuous problem (4.2.6) is guaranteed by Lemma 4.2.4.

Lemma 4.4.2. *The discrete problem (4.3.1) is well-posed for $\alpha < 0$, provided $-\alpha$ is not a Maxwell eigenvalue and h is sufficiently small. Under these conditions the estimate (4.4.4) remains valid.*

Proof. First we establish an *a priori* estimate. Assume that the solution ξ_h of the discrete problem (4.3.1) exists. Then we apply the same duality argument as in the proof of Lemma 4.4.1 to obtain the estimate (cf. (4.4.10))

$$\|\xi - \xi_h\|_{L_2(\Omega)} \leq C h^\beta |\xi - \xi_h|_{H^1(\Omega)}. \qquad (4.4.11)$$

Let $v \in V_h$. It follows from (4.4.5) and (4.4.7) that

$$
\begin{aligned}
|\xi - \xi_h|_{H^1(\Omega)}^2 + \alpha \|\xi - \xi_h\|_{L_2(\Omega)}^2 &= (\nabla \times (\xi - \xi_h), \nabla \times (\xi - v)) + \alpha(\xi - \xi_h, \xi - v) \\
&\leq C |\xi - \xi_h|_{H^1(\Omega)} |\xi - v|_{H^1(\Omega)},
\end{aligned}
$$

which together with the estimate (4.4.11) implies that

$$|\xi - \xi_h|^2_{H^1(\Omega)} \leq C(|\xi - \xi_h|_{H^1(\Omega)}|\xi - v|_{H^1(\Omega)} + h^{2\beta}|\xi - \xi_h|^2_{H^1(\Omega)}). \tag{4.4.12}$$

Hence, for h sufficiently small, we have

$$|\xi - \xi_h|_{H^1(\Omega)} \leq C|\xi - v|_{H^1(\Omega)} \quad \forall v \in V_h, \tag{4.4.13}$$

which together with (4.4.11) implies the estimate (4.4.4).

If ξ_h is the solution of (4.3.1) corresponding to $\boldsymbol{f} = \mathbf{0}$, then $\xi = 0$ is a solution of (4.3.1) and it follows from (4.4.4) that $\xi_h = 0$ (let $v = 0$ in (4.4.4)). Hence the homogeneous discrete problem has a unique solution and, since V_h is finite dimensional, this implies that the discrete problem (4.3.1) is well-posed for h sufficiently small. □

Under the assumption $\boldsymbol{f} \in [L_2(\Omega)]^2$, we have the following stability estimate from the well-posedness of the continuous problem:

$$\|\xi\|_{H^1(\Omega)} \leq C\|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{4.4.14}$$

which together with (4.4.4) immediately implies the following corollary.

Corollary 4.4.3. *Under the assumptions in Lemmas 4.4.1 and 4.4.2, we have*

$$\|\xi - \xi_h\|_{L_2(\Omega)} \leq Ch^\beta\|\boldsymbol{f}\|_{L_2(\Omega)}.$$

Next we compare ϕ_h and ϕ.

Lemma 4.4.4. *For h sufficiently small, we have*

$$\|\nabla \times (\phi - \phi_h)\|_{L_2(\Omega)} \leq C(h^\beta \inf_{v \in V_h} \|\nabla \times (\xi - v)\|_{L_2(\Omega)} + \inf_{v \in V_h} \|\nabla \times (\phi - v)\|_{L_2(\Omega)}). \tag{4.4.15}$$

Proof. Since $(\xi, 1) = 0$, we can define $\tilde{\phi}_h \in V_h$ to be the unique solution of

$$(\nabla \times \tilde{\phi}_h, \nabla \times v) = (\xi, v) \quad \forall v \in V_h, \tag{4.4.16a}$$

$$(\tilde{\phi}_h, 1) = 0. \tag{4.4.16b}$$

It follows from (4.3.4) and (4.4.16) that

$$(\nabla \times (\tilde{\phi}_h - \phi_h), \nabla \times v) = (\xi - \xi_h, v) \quad \forall v \in V_h, \tag{4.4.17}$$

and $(\phi_h - \tilde{\phi}_h, 1) = 0$. We then obtain, by (4.3.3), (4.4.4) and (4.4.17),

$$\begin{aligned}
\|\nabla \times (\tilde{\phi}_h - \phi_h)\|^2_{L_2(\Omega)} &= (\xi - \xi_h, \tilde{\phi}_h - \phi_h) \\
&\leq C\|\xi - \xi_h\|_{L_2(\Omega)} \|\tilde{\phi}_h - \phi_h\|_{L_2(\Omega)} \\
&\leq Ch^\beta \inf_{v \in V_h} \|\nabla \times (\xi - v)\|_{L_2(\Omega)} \|\nabla \times (\tilde{\phi}_h - \phi_h)\|_{L_2(\Omega)},
\end{aligned}$$

which implies

$$\|\nabla \times (\tilde{\phi}_h - \phi_h)\|_{L_2(\Omega)} \leq Ch^\beta \inf_{v \in V_h} \|\nabla \times (\xi - v)\|_{L_2(\Omega)}. \tag{4.4.18}$$

Comparing (4.2.2) and (4.4.16a), we have the Galerkin orthogonality

$$(\nabla \times (\phi - \tilde{\phi}_h), \nabla \times v) = 0 \quad \forall v \in V_h,$$

which implies that, for $v \in V_h$,

$$\begin{aligned}
\|\nabla \times (\phi - \tilde{\phi}_h)\|^2_{L_2(\Omega)} &= (\nabla \times (\phi - \tilde{\phi}), \nabla \times (\phi - v)) \\
&\leq \|\nabla \times (\phi - \tilde{\phi}_h)\|_{L_2(\Omega)} \|\nabla \times (\phi - v)\|_{L_2(\Omega)}
\end{aligned}$$

and hence

$$\|\nabla \times (\phi - \tilde{\phi}_h)\|_{L_2(\Omega)} \leq \inf_{v \in V_h} \|\nabla \times (\phi - v)\|_{L_2(\Omega)}. \tag{4.4.19}$$

Since $\tilde{\phi}_h \in V_h$, the estimate (4.4.19) implies

$$\|\nabla \times (\phi - \tilde{\phi}_h)\|_{L_2(\Omega)} = \inf_{v \in V_h} \|\nabla \times (\phi - v)\|_{L_2(\Omega)}. \tag{4.4.20}$$

The estimate (4.4.15) follows from (4.4.18) and (4.4.20). $\square$

Note that (4.2.2), (4.2.3) and (4.4.3) imply

$$\inf_{v \in V_h} \|\boldsymbol{\nabla} \times (\phi - v)\|_{L_2(\Omega)} \leq \|\boldsymbol{\nabla} \times (\phi - \Pi_h \phi)\|_{L_2(\Omega)} \leq C h^{\beta} \|\xi\|_{L_2(\Omega)}. \tag{4.4.21}$$

Hence, under the assumption that $\boldsymbol{f} \in [L_2(\Omega)]^2$, we can use (4.4.14), (4.4.15) and (4.4.21) to obtain the following bound:

$$\|\boldsymbol{\nabla} \times (\phi - \phi_h)\|_{L_2(\Omega)} \leq C h^{\beta} \|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.4.22}$$

The next result follows from a standard argument using (4.4.2) and Galerkin orthogonality.

Lemma 4.4.5. *We have, for* $1 \leq i \leq p$,

$$|\varphi_i - \varphi_{i,h}|_{H^1(\Omega)} \leq C h^{\beta}. \tag{4.4.23}$$

Proof. Combining the weak formulation of (4.2.11) for φ_i and (4.3.4), we have the Galerkin orthogonality

$$(\boldsymbol{\nabla}(\varphi_i - \varphi_{i,h}), \boldsymbol{\nabla} v) = 0 \quad \forall v \in V_h \cap H_0^1(\Omega) \tag{4.4.24}$$

and hence

$$\begin{aligned} |\varphi_i - \varphi_{i,h}|_{H^1(\Omega)}^2 &= (\boldsymbol{\nabla}(\varphi_i - \varphi_{i,h}), \boldsymbol{\nabla}(\varphi_i - v)) \\ &\leq |\varphi_i - \varphi_{i,h}|_{H^1(\Omega)} |\varphi_i - v|_{H^1(\Omega)} \quad \forall v \in V_h \cap H_0^1(\Omega), \end{aligned}$$

which implies that

$$|\varphi_i - \varphi_{i,h}|_{H^1(\Omega)} \leq \inf_{v \in V_h \cap H_0^1(\Omega)} |\varphi_i - v|_{H^1(\Omega)} \leq |\varphi_i - \Pi_h \varphi_i|_{H^1(\Omega)}. \tag{4.4.25}$$

By the interpolation error estimate (cf. Section 2.4) and (4.4.25), we have the estimate (4.4.23). $\square$

Now we compare $c_{i,h}$ and c_i.

Lemma 4.4.6. *For h sufficiently small, we have*

$$|c_i - c_{i,h}| \le Ch^{\beta}\|\boldsymbol{f}\|_{L_2(\Omega)} \quad \textit{for} \quad 1 \le i \le p. \tag{4.4.26}$$

Proof. First we observe that (4.4.23) implies

$$|(\boldsymbol{f}, \boldsymbol{\nabla}\varphi_i) - (\boldsymbol{f}, \boldsymbol{\nabla}\varphi_{i,h})| \le Ch^{\beta}\|\boldsymbol{f}\|_{L_2(\Omega)} \quad \text{for } 1 \le i \le p. \tag{4.4.27}$$

Furthermore, since $\varphi_i - \varphi_{i,h} \in H_0^1(\Omega)$ for $1 \le i \le p$, (4.2.12) implies that, for $1 \le i, j \le p$,

$$\begin{aligned}
(\boldsymbol{\nabla}\varphi_i, \boldsymbol{\nabla}\varphi_j) - (\boldsymbol{\nabla}\varphi_{i,h}, \boldsymbol{\nabla}\varphi_{j,h}) &= (\boldsymbol{\nabla}\varphi_i, \boldsymbol{\nabla}\varphi_j) + (\boldsymbol{\nabla}(\varphi_{i,h} - \varphi_i), \boldsymbol{\nabla}\varphi_j) \\
&\quad - (\boldsymbol{\nabla}\varphi_{i,h}, \boldsymbol{\nabla}\varphi_{j,h}) + (\boldsymbol{\nabla}\varphi_i, \boldsymbol{\nabla}(\varphi_{j,h} - \varphi_j)) \\
&= (\boldsymbol{\nabla}(\varphi_i - \varphi_{i,h}), \boldsymbol{\nabla}(\varphi_{j,h} - \varphi_j))
\end{aligned}$$

and hence, in view of (4.4.23),

$$|(\boldsymbol{\nabla}\varphi_i, \boldsymbol{\nabla}\varphi_j) - (\boldsymbol{\nabla}\varphi_{i,h}, \boldsymbol{\nabla}\varphi_{j,h})| \le Ch^{2\beta} \quad \text{for } 1 \le i, j \le p. \tag{4.4.28}$$

We can rewrite (4.2.14) and (4.3.6) as

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{b} \quad \text{and} \quad \boldsymbol{A}_h\boldsymbol{c}_h = \boldsymbol{b}_h,$$

where $\boldsymbol{c} \in \mathbb{R}^p$ (resp. $\boldsymbol{c}_h \in \mathbb{R}^p$) is the vector whose j-th component is c_j (resp. $c_{j,h}$), $\boldsymbol{A} \in \mathbb{R}^{p\times p}$ (resp. $\boldsymbol{A}_h \in \mathbb{R}^{p\times p}$) is the matrix whose (i,j)-th component is $(\boldsymbol{\nabla}\varphi_j, \boldsymbol{\nabla}\varphi_i)$ (resp. $(\boldsymbol{\nabla}\varphi_{j,h}, \boldsymbol{\nabla}\varphi_{i,h})$), and $\boldsymbol{b} \in \mathbb{R}^p$ (resp. $\boldsymbol{b}_h \in \mathbb{R}^p$) is the vector whose j-th component is $\alpha^{-1}(\boldsymbol{f}, \boldsymbol{\nabla}\varphi_j)$ (resp. $\alpha^{-1}(\boldsymbol{f}, \boldsymbol{\nabla}\varphi_{j,h})$).

Note that

$$\|\boldsymbol{b}\|_\infty \le |\alpha|^{-1}(\max_{1\le i\le p}\|\boldsymbol{\nabla}\varphi_i\|_{L_2(\Omega)})\|\boldsymbol{f}\|_{L_2(\Omega)} \le C\|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{4.4.29}$$

and the estimates (4.4.27)-(4.4.28) are translated into

$$\|\boldsymbol{b} - \boldsymbol{b}_h\|_\infty \le Ch^{\beta}\|\boldsymbol{f}\|_{L_2(\Omega)} \quad \text{and} \quad \|\boldsymbol{A} - \boldsymbol{A}_h\|_\infty \le Ch^{2\beta}. \tag{4.4.30}$$

The estimate (4.4.26) follows from the identity

$$\boldsymbol{c}-\boldsymbol{c}_h=\boldsymbol{A}^{-1}\boldsymbol{b}-\boldsymbol{A}_h^{-1}\boldsymbol{b}_h=\boldsymbol{A}^{-1}(\boldsymbol{b}-\boldsymbol{b}_h)+\boldsymbol{A}^{-1}(\boldsymbol{A}_h-\boldsymbol{A})\boldsymbol{A}_h^{-1}((\boldsymbol{b}_h-\boldsymbol{b})+\boldsymbol{b})$$

and (4.4.29)-(4.4.30). □

Putting all the lemmas together, we can deduce the following error estimate for $\mathring{\boldsymbol{u}}_h$.

Theorem 4.4.7. *When $\alpha \geq 0$ in (4.1.1), we have*

$$\|\mathring{\boldsymbol{u}}-\mathring{\boldsymbol{u}}_h\|_{L_2(\Omega)} \leq Ch^{\beta}\|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{4.4.31}$$

where β is defined by (4.4.1). When $\alpha < 0$ and h sufficiently small, we also have (4.4.31).

Proof. From (4.4.29), we have that the solutions $c_1, c_2, \cdots, c_p$ of (4.2.14) satisfy

$$|c_i| \leq C\|\boldsymbol{f}\|_{L_2(\Omega)} \quad \text{for } 1 \leq i \leq p. \tag{4.4.32}$$

From (4.2.1), (4.2.14) and (4.3.7), we have that

$$\begin{aligned}
\|\mathring{\boldsymbol{u}}-\mathring{\boldsymbol{u}}_h\|_{L_2(\Omega)} \leq & C|\phi-\phi_h|_{H^1(\Omega)}+\sum_{i=1}^{p}|c_i\varphi_i-c_{i,h}\varphi_{i,h}|_{H^1(\Omega)} \\
\leq & C|\phi-\phi_h|_{H^1(\Omega)}+\sum_{i=1}^{p}(|c_i-c_{i,h}||\varphi_i|_{H^1(\Omega)}+|c_{i,h}||\varphi_i-\varphi_{i,h}|_{H^1(\Omega)})
\end{aligned} \tag{4.4.33}$$

$$\begin{aligned}
\leq & C|\phi-\phi_h|_{H^1(\Omega)}+\sum_{i=1}^{p}|c_i-c_{i,h}|(|\varphi_i|_{H^1(\Omega)}+|\varphi_i-\varphi_{i,h}|_{H^1(\Omega)}) \\
& +\sum_{i=1}^{p}|c_i||\varphi_i-\varphi_{i,h}|_{H^1(\Omega)}.
\end{aligned}$$

The estimate (4.4.31) follows from (4.4.22), (4.4.23), (4.4.26), (4.4.32) and (4.4.33).

□

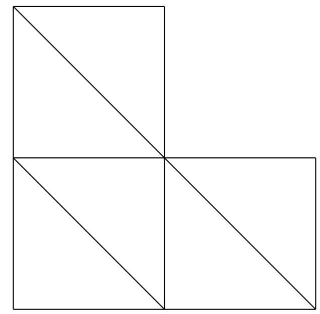

FIGURE 4.1. A uniform mesh on the L-shaped domain

Remark 4.4.8. In the case where $c_i = 0 = c_{i,h}$ for $1 \le i \le p$, it follows from (4.4.15) and (4.4.33) that

$$\|\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h\|_{L_2(\Omega)} \le C(h^\beta \inf_{v\in V_h} \|\boldsymbol{\nabla} \times (\xi - v)\|_{L_2(\Omega)} + \inf_{v\in V_h} \|\boldsymbol{\nabla} \times (\phi - v)\|_{L_2(\Omega)}).$$

4.5 Numerical Results

In this section we present the results of several numerical experiments that confirm the theoretical results obtained in Section 4.4.

In the first set of experiments, we examine the convergence behavior of the numerical scheme on the L-shaped domain $(-1,1)^2 \setminus [0,1]^2$ with uniform meshes (See Figure 4.1). The exact solution is chosen to be

$$\mathring{\boldsymbol{u}} = \boldsymbol{\nabla} \times (r^{2/3} \cos(\frac{2}{3}\theta - \frac{\pi}{3})\Phi(x)), \tag{4.5.1}$$

where (r, θ) are the polar coordinates at the origin and $\Phi(x) = (1 - x_1^2)^2(1 - x_2^2)^2$. It has the correct Maxwell singularity at the reentrant corner. We solve (4.1.1) for $\alpha = -1$, 0 and 1, with $\boldsymbol{f} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}) + \alpha\mathring{\boldsymbol{u}} \in H(div^0; \Omega)$. The results are tabulated in Table 4.1.

Note that the convergence of $\mathring{\boldsymbol{u}}_h$ to $\mathring{\boldsymbol{u}}$ is approaching the order of $\beta = 2/3$, which is predicted by Theorem 4.4.7. On the other hand, since $\xi = \boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}$ behaves like $r^{2/3}$ at the origin, the order of convergence for ξ_h according to (4.4.4) is $(2/3) + (2/3) = 4/3$, which agrees with the observed order of convergence.

h	$\frac{\|\nabla\times\mathring{\boldsymbol{u}}-\xi_h\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order	h	$\frac{\|\mathring{\boldsymbol{u}}-\mathring{\boldsymbol{u}}_{\boldsymbol{h}}\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order
$\alpha=-1$					
$1/8$	3.57E-02	1.43	$1/8$	3.19E-02	1.41
$1/16$	1.32E-02	1.43	$1/16$	1.23E-02	1.38
$1/32$	4.98E-03	1.41	$1/32$	5.03E-03	1.28
$1/64$	1.90E-03	1.39	$1/64$	2.26E-03	1.15
$1/128$	7.37E-04	1.37	$1/128$	1.13E-03	0.99
$1/256$	2.87E-04	1.36	$1/256$	6.17E-04	0.87
$\alpha=0$					
$1/8$	1.12E-02	1.44	$1/8$	1.35E-02	1.29
$1/16$	4.24E-03	1.41	$1/16$	6.13E-03	1.14
$1/32$	1.63E-03	1.38	$1/32$	3.07E-03	0.99
$1/64$	6.36E-04	1.36	$1/64$	1.66E-03	0.89
$1/128$	2.50E-04	1.35	$1/128$	9.46E-04	0.81
$1/256$	9.86E-05	1.34	$1/256$	5.58E-04	0.76
$\alpha=1$					
$1/8$	6.77E-03	1.39	$1/8$	1.06E-02	1.14
$1/16$	2.63E-03	1.36	$1/16$	5.27E-03	1.01
$1/32$	1.04E-03	1.34	$1/32$	2.80E-03	0.91
$1/64$	4.14E-04	1.33	$1/64$	1.56E-03	0.84
$1/128$	1.65E-04	1.33	$1/128$	9.06E-04	0.79
$1/256$	6.57E-05	1.32	$1/256$	5.38E-04	0.75

TABLE 4.1. Results for (4.1.1) on the L-shaped domain with exact solution given by (4.5.1)

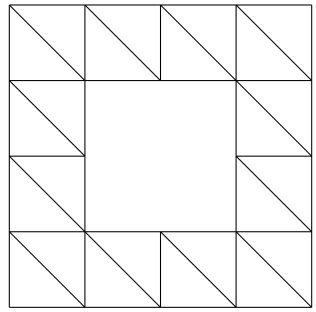

FIGURE 4.2. A uniform mesh on the doubly connected domain

In the second set of experiments, we examine the convergence behavior of the numerical scheme on the doubly connected domain $(0,4)^2 \setminus [1,3]^2$ with uniform meshes (See Figure 4.2).

In this case the solution $\mathring{\boldsymbol{u}}$ of (4.1.1) can be written as

$$\mathring{\boldsymbol{u}} = \boldsymbol{\nabla} \times \phi + c\boldsymbol{\nabla}\varphi, \tag{4.5.2}$$

where c is a constant and the harmonic function φ satisfies the boundary conditions

$$\varphi|_{\Gamma_0} = 0 \quad \text{and} \quad \varphi|_{\Gamma_1} = 1.$$

Here Γ_0 (resp. Γ_1) is the boundary of $(0,4)^2$ (resp. $(1,3)^2$). The exact solution is chosen to be

$$\mathring{\boldsymbol{u}} = \begin{bmatrix} x_2(1-x_2)(3-x_2)(4-x_2) \\ x_1(1-x_1)(3-x_1)(4-x_1) \end{bmatrix}. \tag{4.5.3}$$

We solve (4.1.1) for $\alpha = -1$ and 1, with $\boldsymbol{f} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}) + \alpha\mathring{\boldsymbol{u}} \in H(div^0;\Omega)$. The results are tabulated in Table 4.2.

Note that in this case $\mathring{\boldsymbol{u}}$ is the curl of a quintic polynomial and hence $c = 0$ in (4.5.2). In fact, since $\boldsymbol{f}$ is also the curl of a polynomial, we have $(\boldsymbol{f}, \boldsymbol{\nabla}\varphi_h) = 0$ by Lemma 3.2.9, and it is observed that $c_h = 0$ up to machine error.

According to Remark 4.4.8, the order of convergence for $\mathring{\boldsymbol{u}}_h$ is 1 (since ξ and ϕ are smooth), which is observed. The order of convergence for ξ_h is found to be 2,

h	$\frac{\|\nabla\times\mathring{\boldsymbol{u}}-\xi_h\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order	$\|c\|_h$	$\frac{\|\mathring{\boldsymbol{u}}-\mathring{\boldsymbol{u}}_{\boldsymbol{h}}\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order
$\alpha=-1$					
1/8	3.71E-03	2.01	7.93E-17	1.13E-02	1.05
1/16	9.26E-04	2.00	1.36E-16	5.61E-03	1.01
1/32	2.31E-04	2.00	1.49E-16	2.80E-03	1.00
1/64	5.78E-05	2.00	7.69E-16	1.39E-03	1.00
1/128	1.44E-05	2.00	7.43E-16	6.99E-04	1.00
$\alpha=1$					
1/8	1.69E-03	1.98	9.25E-16	9.50E-03	1.00
1/16	4.25E-04	1.99	1.11E-15	4.75E-03	1.00
1/32	1.06E-04	2.00	1.35E-15	2.38E-03	1.00
1/64	2.66E-05	2.00	3.27E-15	1.19E-03	1.00
1/128	6.64E-06	2.00	4.96E-15	5.94E-04	1.00

TABLE 4.2. Results for (4.1.1) on the doubly connected domain with exact solution given by (4.5.3)

which is better than the order of $\beta+1=5/3$ predicted by (4.4.4). This is likely due to the effects of superconvergence since we use uniform meshes in computing ξ_h and the exact solution ξ is smooth.

Finally we take the right-hand side of (4.1.1) to be the piecewise smooth vector field

$$\boldsymbol{f}=\begin{cases}\begin{bmatrix}1+x_1\\ 0\end{bmatrix} & \text{if } x_1<x_2 \quad \text{and} \quad 3<x_1<4,\\ \begin{bmatrix}0\\ 1+x_2\end{bmatrix} & \text{otherwise.}\end{cases} \tag{4.5.4}$$

The results are tabulated in Table 4.3.

The observed orders of convergence are consistent with the theoretical results. In particular, the order of convergence for $\mathring{\boldsymbol{u}}_h$ is $2/3$ for $\alpha=1$ and approaching $2/3$ for $\alpha=-1$, which agrees with the estimate (4.4.31). The order of convergence for c_h is $2/3+1/2=7/6$. This is because $\boldsymbol{f}$ is piecewise smooth and hence the estimate (4.4.26) can be improved (cf. [14, Remark 4.8]). The order of convergence for ξ_h in

h	$\frac{\|\nabla\times\hat{\boldsymbol{u}}-\xi_h\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order	$\|c_h\|$	Order	$\frac{\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_h\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order
$\alpha=-1$						
1/8	1.72E-01	1.26	0.763918	1.05	2.68E-01	1.00
1/16	5.28E-02	1.70	0.765285	0.87	1.28E-01	1.06
1/32	1.49E-02	1.83	0.765991	0.95	6.93E-02	0.89
1/64	4.29E-03	1.80	0.766332	1.05	4.04E-02	0.78
1/128	1.13E-03	1.69	0.766489	1.12	2.42E-02	0.73
$\alpha=1$						
1/8	1.03E-02	1.33	-0.763918	1.05	8.60E-02	0.71
1/16	4.04E-03	1.35	-0.765285	0.87	5.30E-02	0.70
1/32	1.58E-03	1.35	-0.765991	0.95	3.29E-02	0.69
1/64	6.21E-04	1.35	-0.766332	1.05	2.05E-02	0.68
1/128	2.44E-04	1.34	-0.766489	1.12	1.28E-02	0.67

TABLE 4.3. Results for (4.1.1) on the doubly connected domain with right-hand side given by (4.5.4)

both cases is higher than the order predicted by (4.4.3). This is probably due to the fact that the mesh size h is not small enough and the asymptotic behavior has not been reached.

Chapter 5

Multigrid Methods for Maxwell Equations in Heterogeneous Media

5.1 Introduction

Let Ω be a bounded simply connected polygonal domain in $\mathbb{R}^2$, and Ω_j, $1 \le j \le J$, be open polygonal subdomains of Ω that form a partition of Ω, i.e.,

$$\Omega_{j1} \cap \Omega_{j2} = \emptyset \quad \text{for } j_1 \neq j_2 \quad \text{and } \cup_{j=1}^{J} \bar{\Omega}_j = \bar{\Omega}.$$

Let $\Gamma = \cup \Gamma_{j_1 j_2}$ be the interface of Ω, where $\Gamma_{j_1 j_2} = \bar{\Omega}_{j_1} \cap \bar{\Omega}_{j_2}$, if $\bar{\Omega}_{j_1} \cap \bar{\Omega}_{j_2} \neq \emptyset$.

Let $\boldsymbol{f} \in [L_2(\Omega)]^2$ and ϵ, μ be piecewise constant functions in the domain Ω such that $\epsilon(x) = \epsilon_j$ and $\mu(x) = \mu_j$ for $x \in \Omega_j$, with the assumption that ϵ_j and μ_j are positive numbers.

We will consider the following weak formulation of the Maxwell interface problem:

Find $\mathring{\boldsymbol{u}} \in H_0(curl;\Omega) \cap H(div^0;\Omega;\epsilon)$ such that

$$(\mu^{-1}\boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}, \boldsymbol{\nabla} \times \boldsymbol{v}) + \alpha(\epsilon\mathring{\boldsymbol{u}}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in H_0(curl;\Omega) \cap H(div^0;\Omega;\epsilon), \tag{5.1.1}$$

where the space $H(div^0;\Omega;\epsilon)$ is defined in Subsection 3.2.2.

Remark 5.1.1. The variational formulation here actually implies the interface conditions

$$[\boldsymbol{n} \times \mathring{\boldsymbol{u}}] = \boldsymbol{n}_- \times \boldsymbol{u}_- + \boldsymbol{n}_+ \times \boldsymbol{u}_+ = \boldsymbol{0}$$

and

$$[\boldsymbol{n} \cdot (\epsilon\mathring{\boldsymbol{u}})] = \boldsymbol{n}_- \cdot (\epsilon_- \boldsymbol{u}_-) + \boldsymbol{n}_+ \cdot (\epsilon_+ \boldsymbol{u}_+) = 0$$

in the sense of $H^{-1/2}(\Gamma)$, where Γ is the interface. The first interface condition above comes from the assumption that $\mathring{\boldsymbol{u}} \in H(curl;\Omega)$. The second interface condition above means that there is no charge density distribution on the interface. It comes from the assumption that $\mathring{\boldsymbol{u}} \in H(div^0; \Omega; \epsilon)$.

Under the assumption that Ω is simply connected, we can write (cf. Lemma 3.2.12)

$$\epsilon \mathring{\boldsymbol{u}} = \nabla \times \phi, \tag{5.1.2}$$

where $\phi \in H^1(\Omega)$ satisfies $(\phi, 1) = 0$. Then we can show (cf. Section 5.2) that the function ϕ in (5.1.2) is determined by

$$(\nabla \times \phi, \epsilon^{-1} \nabla \times \psi) = (\mu\xi, \psi) \quad \forall \psi \in H^1(\Omega) \tag{5.1.3}$$

and the constraint

$$(\phi, 1) = 0, \tag{5.1.4}$$

where the function $\xi = \mu^{-1} \nabla \times \mathring{\boldsymbol{u}} \in H^1(\Omega)$ is determined by

$$(\nabla \times \xi, \epsilon^{-1} \nabla \times \psi) + \alpha(\mu\xi, \psi) = (\boldsymbol{f}, \epsilon^{-1} \nabla \times \psi) \quad \forall \psi \in H^1(\Omega), \tag{5.1.5}$$

when $\alpha \neq 0$ and by the equation (5.1.5) with the constraint

$$(\mu\xi, 1) = 0, \tag{5.1.6}$$

when $\alpha = 0$.

We can therefore solve (5.1.1) by the following numerical procedure under the assumption that $-\alpha$ is not a Maxwell eigenvalue.

Step 1. Compute a numerical approximation $\tilde{\xi}$ of ξ by solving the interface problem (5.1.5) when $\alpha \neq 0$, and by solving (5.1.5) with the constraint $(\xi, 1) = 0$ when $\alpha = 0$.

Step 2. Compute a numerical approximation $\tilde{\phi}$ of ϕ by solving the interface problem (5.1.3) under the constraint $(\phi, 1) = 0$, where ξ is replaced by $\tilde{\xi}$.

Step 3. The numerical approximation $\tilde{\boldsymbol{u}}$ for $\mathring{\boldsymbol{u}}$ is given by

$$\tilde{\boldsymbol{u}} = \boldsymbol{\nabla} \times \tilde{\phi}.$$

Because equations (5.1.5) and (5.1.3) are elliptic interface problems, we already know that the solutions of these equations have very low regularity (cf. Subsection 2.2.2) and hence the solution of Maxwell's interface problem (5.1.1) can have very low regularity. Therefore the P_1 finite element method does not work well in this case. However we can take advantage of the singular function representations of elliptic interface problems and the extraction formulas for the stress intensity factors (cf. the discussion in Subsection 2.3.2) to recover the optimal convergence of the P_1 finite element method on quasi-uniform grids by using a full multigrid approach for the interface problems (5.1.5) and (5.1.3).

5.2 Equation for ξ and ϕ

Our goal in this section is to justify the equations (5.1.3)–(5.1.6).

Lemma 5.2.1. *Given $\boldsymbol{v} \in H(div^0;\Omega;\epsilon)$, there exists a unique $\phi \in H^1(\Omega)$ such that $(\phi, 1) = 0$ and*

$$\epsilon \boldsymbol{v} = \nabla \times \phi.$$

Proof. By Lemma 3.2.12, we have

$$\boldsymbol{v} = \epsilon^{-1} \nabla \times \varphi,$$

for some $\varphi \in H^1(\Omega)$. Let $\phi = \varphi - \frac{1}{|\Omega|} \int_\Omega \varphi dx$, then it satisfies the constraint $(\phi, 1) = 0$. The uniqueness comes from Friedrichs' inequality (cf. [20, Lemma 4.3.14]). $\square$

Now we use Lemma 5.2.1 to justify the equation (5.1.3). For any $\psi \in H^1(\Omega)$ we have

$$\begin{aligned}(\boldsymbol{\nabla} \times \phi, \epsilon^{-1} \boldsymbol{\nabla} \times \psi) &= (\epsilon \mathring{\boldsymbol{u}}, \epsilon^{-1} \boldsymbol{\nabla} \times \psi) = (\mathring{\boldsymbol{u}}, \boldsymbol{\nabla} \times \psi) \\ &= (\boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}, \psi) = (\mu \xi, \psi),\end{aligned}$$

since $\mathring{\boldsymbol{u}} \in H_0(curl;\Omega)$ satisfies Lemma 3.1.12. To justify the equation (5.1.5), we need another lemma (cf. [14]).

Lemma 5.2.2. *The solution $\mathring{\boldsymbol{u}}$ of (5.1.1) satisfies*

$$\boldsymbol{\nabla} \times (\mu^{-1} \boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}) + \alpha(\epsilon \mathring{\boldsymbol{u}}) = \epsilon Q(\epsilon^{-1} \boldsymbol{f}) \tag{5.2.1}$$

in the sense of distributions, where $Q : [L_2(\Omega;\epsilon)]^2 \to H(div^0;\Omega;\epsilon)$ is the orthogonal projection.

Proof. Let $\boldsymbol{\zeta} \in [\mathcal{D}(\Omega)]^2$ be a C^∞ vector field with compact support in Ω. So $\boldsymbol{\zeta} \in H_0(curl;\Omega)$, $Q\boldsymbol{\zeta} \in H(div^0;\Omega;\epsilon)$ and $\boldsymbol{\zeta} - Q\boldsymbol{\zeta} \in \nabla H_0^1(\Omega)$ because of Theorem 3.2.7.

Because of Corollary 3.2.10, we have $\boldsymbol{\zeta} - Q\boldsymbol{\zeta} \in H_0(curl;\Omega)$ and hence $Q\boldsymbol{\zeta} \in H_0(curl;\Omega)$. Therefore

$$Q\boldsymbol{\zeta} \in H_0(curl;\Omega) \cap H(div^0;\Omega;\epsilon). \tag{5.2.2}$$

Furthermore, we have

$$\boldsymbol{\nabla} \times (\boldsymbol{\zeta} - Q\boldsymbol{\zeta}) = 0, \tag{5.2.3}$$

since $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} H_0^1(\Omega)) = \{0\}$, and for $\mathring{\boldsymbol{u}} \in H(div^0;\Omega;\epsilon)$, we have

$$(\epsilon \mathring{\boldsymbol{u}}, \boldsymbol{\zeta} - Q\boldsymbol{\zeta}) = 0. \tag{5.2.4}$$

Using (5.1.1), (5.2.2), (5.2.3) and (5.2.4), we have

$$\begin{aligned}
(\mu \boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}, \boldsymbol{\nabla} \times \boldsymbol{\zeta}) + \alpha(\epsilon \mathring{\boldsymbol{u}}, \boldsymbol{\zeta}) =& (\mu \boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}, \boldsymbol{\nabla} \times (Q\boldsymbol{\zeta} + \boldsymbol{\zeta} - Q\boldsymbol{\zeta})) \\
& + \alpha(\epsilon \mathring{\boldsymbol{u}}, Q\boldsymbol{\zeta} + (\boldsymbol{\zeta} - Q\boldsymbol{\zeta})) \\
=& (\mu \boldsymbol{\nabla} \times \mathring{\boldsymbol{u}}, \boldsymbol{\nabla} \times Q\boldsymbol{\zeta}) + \alpha(\epsilon \mathring{\boldsymbol{u}}, Q\boldsymbol{\zeta}) \\
=& (\boldsymbol{f}, Q\boldsymbol{\zeta}) \\
=& (Q(\epsilon^{-1} \boldsymbol{f}), \boldsymbol{\zeta})_{L_2(\Omega;\epsilon)}
\end{aligned}$$

$$=(\epsilon Q(\epsilon^{-1}\boldsymbol{f}), \boldsymbol{\zeta}),$$

which completes the proof. $\square$

Remark 5.2.3. Lemma 5.2.2 is a generalization of Lemma 4.2.2.

With the help of Lemma 3.2.2, equation (5.1.5) can be justified by the same argument as in Chapter 4.

Let $\psi \in H^1(\Omega)$ be arbitrary. Since $\epsilon^{-1}\boldsymbol{\nabla}\times\psi \in H(div^0;\Omega;\epsilon)$ by Theorem 3.2.2 and Definition 3.2.5, we have

$$\begin{aligned}
(\boldsymbol{f}, \epsilon^{-1}\boldsymbol{\nabla}\times\psi) &= (\epsilon^{-1}\boldsymbol{f}, \epsilon^{-1}\boldsymbol{\nabla}\times\psi)_{L_2(\Omega;\epsilon)} \\
&= (Q(\epsilon^{-1}\boldsymbol{f}), \epsilon^{-1}\boldsymbol{\nabla}\times\psi)_{L_2(\Omega;\epsilon)} \\
&= (\epsilon Q(\epsilon^{-1}\boldsymbol{f}), \epsilon^{-1}\boldsymbol{\nabla}\times\psi) \\
&= (\boldsymbol{\nabla}\times\xi + \alpha(\epsilon\mathring{\boldsymbol{u}}), \epsilon^{-1}\boldsymbol{\nabla}\times\psi) \quad \text{(by Lemma 5.2.2)} \\
&= (\boldsymbol{\nabla}\times\xi, \epsilon^{-1}\boldsymbol{\nabla}\times\psi) + \alpha(\boldsymbol{\nabla}\times\mathring{\boldsymbol{u}}, \psi) \quad \text{(by Lemma 3.2.9)} \\
&= (\boldsymbol{\nabla}\times\xi, \epsilon^{-1}\boldsymbol{\nabla}\times\psi) + \alpha(\mu\xi, \psi),
\end{aligned}$$

which gives (5.1.5). The constraint $(\xi, 1) = 0$ follows immediately from Lemma 3.1.12.

5.3 Regularity, Stress Intensity Factors and Extraction Formulas

For simplicity we will assume from here on that there is only one interface vertex p_* of the subdomains near which the solution ϕ of (5.1.3) and/or the solution ξ of (5.1.5) are singular.

We further assume that $\boldsymbol{f} \in H^1(\Omega_j)$ and Ω_j is convex for $1 \leq j \leq J$. Then by integration by parts, the weak problems (5.1.5) and (5.1.3) are equivalent to the following strong problems:

Find $\xi \in H^1(\Omega)$ such that

$$-\epsilon_j^{-1}\Delta\xi + \alpha\mu_j\xi = \epsilon_j^{-1}\boldsymbol{\nabla}\times\boldsymbol{f} \quad \text{in } \Omega_j, 1\le j\le J, \tag{5.3.1a}$$

$$\epsilon^{-1}\frac{\partial\xi}{\partial n} = -\epsilon^{-1}\boldsymbol{n}\times\boldsymbol{f} \quad \text{on the boundary } \partial\Omega, \tag{5.3.1b}$$

$$\left[\epsilon^{-1}\frac{\partial\xi}{\partial n}\right] = -[\epsilon^{-1}\boldsymbol{n}\times\boldsymbol{f}] \quad \text{on the interface } \Gamma, \tag{5.3.1c}$$

and find $\phi \in H^1(\Omega)$ such that

$$-\epsilon_j^{-1}\Delta\phi = \mu_j\xi \quad \text{in } \Omega_j, 1\le j\le J, \tag{5.3.2a}$$

$$\epsilon^{-1}\frac{\partial\phi}{\partial n} = 0 \quad \text{on the boundary } \partial\Omega, \tag{5.3.2b}$$

$$\left[\epsilon^{-1}\frac{\partial\phi}{\partial n}\right] = 0 \quad \text{on the interface } \Gamma. \tag{5.3.2c}$$

We can rewrite the problem (5.3.2) as a weak problem of the form (2.2.8). For the problem (5.3.1), we can find a function U satisfying the boundary and interface conditions and $U|_{\Omega_j} \in H^2(\Omega_j)$ for $1\le j\le J$. By Theorem 3.2.7, there exist $U_{j,1}\in H_0^1(\Omega_j)$ and $U_{j,2}\in H^1(\Omega_j)$ such that

$$\boldsymbol{f}|_{\Omega_j} = \boldsymbol{\nabla}U_{j,1} + \boldsymbol{\nabla}\times U_{j,2},$$

where $U_{j,1}$ is the solution of

$$(\boldsymbol{\nabla}U_{j,1}, \boldsymbol{\nabla}v) = -(\boldsymbol{\nabla}\cdot\boldsymbol{f}, v) \quad \forall v\in H_0^1(\Omega_j).$$

We have $U_{j,1}\in H^2(\Omega_j)$ since Ω_j is convex and $\boldsymbol{\nabla}\cdot\boldsymbol{f}\in L_2(\Omega_j)$. Therefore,$\boldsymbol{\nabla}\times U_{j,2} = \boldsymbol{f}-\boldsymbol{\nabla}U_{j,1}$ is in $H^1(\Omega_{j,1})$ and hence $U_{j,2}$ is in $H^2(\Omega_j)$. On the boundary $\partial\Omega_j$, we have

$$\epsilon^{-1}\boldsymbol{n}\times(\boldsymbol{\nabla}\times U_{j,2}) = \epsilon^{-1}\boldsymbol{n}\times\boldsymbol{f} = -\epsilon^{-1}\frac{\partial\xi}{\partial n},$$

since $U_{j,1}\in H_0^1(\Omega_j)$ implies $\boldsymbol{n}\times\boldsymbol{\nabla}U_{j,1}=\boldsymbol{0}$ on the boundary $\partial\Omega_j$. Now we define $U = U_{j,2}$ for $x\in\Omega_j$ and $1\le j\le J$. The function U satisfies the corresponding boundary and interface conditions and $U|_{\Omega_j} \in H^2(\Omega_j)$ for $1 \leq j \leq J$. So the function $\xi - U$ satisfies homogeneous boundary and interface conditions and hence is the solution of a weak problem described by (2.2.8). According to the discussion in Subsection 2.2.2, we let $\lambda_l = (\sigma_l)^2, l \geq 1$, be the positive eigenvalues of the Sturm-Liouville problem at p_* and the functions Θ_l be the corresponding eigenfunctions. We define the singular functions s_l by the formula

$$s_l(r, \theta) = \varrho_l(r) r^{\sigma_l} \Theta_l(\theta),$$

where ϱ_l is a smooth cut-off function that equals 1 identically near $r = 0$ and vanishes for $r \geq \delta$.

We have the singular function representations (2.2.11)/ (2.2.14) for the solution ξ of (5.1.5):

$$\xi = \sum_{0<\sigma_l<1} \kappa_l^{\xi} s_l + w_{\xi}, \tag{5.3.3}$$

and the solution ϕ of (5.1.3):

$$\phi = \sum_{0<\sigma_l<1} \kappa_l^{\phi} s_l + w_{\phi}, \tag{5.3.4}$$

where

$$w_{\xi}|_{\Omega_j}, w_{\phi}|_{\Omega_j} \in H^2(\Omega_j) \text{ for } 1 \leq j \leq J.$$

Moreover, the solution ξ and ϕ satisfy the following elliptic regularity estimates (cf. Subsection 2.3.2):

$$\sum_{0<\sigma_l<1} |\kappa_l^{\xi}| + \sum_{j=1}^{J} \|w_{\xi}\|_{H^2(\Omega_j)} \leq C \|\boldsymbol{\nabla} \times \boldsymbol{f}\|_{L_2(\Omega)} \tag{5.3.5}$$

and

$$\sum_{0<\sigma_l<1} |\kappa_l^{\phi}| + \sum_{j=1}^{J} \|w_{\phi}\|_{H^2(\Omega_j)} \leq C \|\boldsymbol{\nabla} \times \boldsymbol{f}\|_{L_2(\Omega)}. \tag{5.3.6}$$

The stress intensity factors κ_l^ξ and κ_l^ϕ can be computed by the following extraction formulas (Lemma 2.3.4 and Lemma 2.3.5)

$$\begin{aligned}\kappa_l^\xi = \frac{1}{2\sigma_l}\{&\int_\Omega (\epsilon^{-1}(\nabla \times \boldsymbol{f})s_l^* - \alpha\mu\xi s_l^* + \epsilon^{-1}\xi\Delta s_l^*)dx \\ &- \int_{\partial\Omega} \epsilon^{-1}(\boldsymbol{n} \times \boldsymbol{f})s_l^* ds - \int_\Gamma [\epsilon^{-1}\boldsymbol{n} \times \boldsymbol{f}]s_l^* ds\},\end{aligned} \tag{5.3.7}$$

and

$$\kappa_l^\phi = \frac{1}{2\sigma_l}\int_\Omega (\mu\xi s_l^* + \epsilon^{-1}\phi\Delta s_l^*)dx. \tag{5.3.8}$$

Now we present the basic idea of our algorithm by focusing on (5.1.5) with $\alpha \neq 0$. For (5.1.3) the idea will be essentially the same, so we omit it here. By (5.1.5) and (5.3.3), w_ξ (the regular part of ξ) is the solution of

$$\begin{aligned}(\nabla \times w_\xi, \epsilon^{-1}\nabla \times \psi) + \alpha(\mu w_\xi, \psi) &= (\boldsymbol{f}, \epsilon^{-1}\boldsymbol{\nabla} \times \psi) \\ &+ \sum_{0<\sigma_l<1} \kappa_l^\xi[(\epsilon^{-1}\Delta s_l, \psi) - \alpha(\mu s_l, \psi)] \quad \forall \psi \in H^1(\Omega).\end{aligned} \tag{5.3.9}$$

We can then solve (5.3.9) using a P_1 finite element method and the convergence rate in H^1-norm would be of order $O(h)$ since $w \in H^2(\Omega_j) \quad \text{for} \quad 1 \leq j \leq J$. But of course we do not know the stress intensity factors κ_l^ξ and therefore we consider the following problem instead:

$$\begin{aligned}(\nabla \times \hat{w}, \epsilon^{-1}\nabla \times \psi) + \alpha(\mu \hat{w}, \psi) &= (\boldsymbol{f}, \epsilon^{-1}\boldsymbol{\nabla} \times \psi) \\ &+ \sum_{0<\sigma_l<1} \kappa_{l,k}^{\hat{\xi}}[(\epsilon^{-1}\Delta s_l, \psi) - \alpha(\mu s_l, \psi)] \quad \forall \psi \in H^1(\Omega).\end{aligned} \tag{5.3.10}$$

Here the numbers $\kappa_{l,k}^{\hat{\xi}}$ are the approximate stress intensity factors computed through the extraction formula (5.3.7) where ξ is replaced by an approximation from the previous level. This strategy can be implemented naturally through the full multigrid methodology.

In the resulting algorithm we are really computing the regular part w of the solution and therefore the improvement in the convergence rate is possible because

w has better regularity than ξ. Actually, the optimal convergence rate of the standard P_1 finite element is recovered, i.e., the convergence rate of w_k to w in the H^1-norm will be $O(h)$ (cf. Theorem 5.5.3).

5.4 The Algorithm

Consider a sequence of triangulations $\{\mathscr{T}_1, \cdots, \mathscr{T}_N\}$ of Ω, where the triangulations are aligned with the interface between subdomains. Suppose $\mathscr{T}_1$ is given and let $\mathscr{T}_k, \quad k \geq 2$, be obtained from $\mathscr{T}_{k-1}$ via a regular subdivision, i.e., edge midpoints in $\mathscr{T}_{k-1}$ are connected by new edges to form $\mathscr{T}_k$. Let V_k be the P_1 finite element space associated with $\mathscr{T}_k$ (cf. Section 2.4), and $\tilde{V}_k$ be the subspace of V_k such that $v_k \in \tilde{V}_k$ iff $(\mu v_k, 1) = 0$. Let $h_k = \max_{T \in \mathscr{T}_k} \text{diam}\, T$. We introduce a discrete inner product $(\cdot,\cdot)_k$ on V_k by

$$(v_1, v_2)_k = h_k^2 \sum v_1(p) v_2(p) \quad \forall v_1, v_2 \in V_k, \tag{5.4.1}$$

where the summation is taken over all the vertices p of $\mathscr{T}_k$.

The operators $M_k : V_k \to V_k$, $A_k : V_k \to V_k$, $I_k^{k-1} : V_k \to V_{k-1}$, $Q_{a,k} : V_k \to \tilde{V}_k$, where $a = \mu$ or 1, are defined by

$$(A_k v_1, v_2)_k = \int_\Omega \epsilon^{-1} \nabla \times v_1 \cdot \nabla \times v_2 dx \quad \forall v_1, v_2 \in V_k, \tag{5.4.2}$$

$$(I_k^{k-1} v, w)_{k-1} = (v, w)_k \quad \forall v \in V_k, w \in V_{k-1} (\subset V_k), \tag{5.4.3}$$

$$(M_k v_1, v_2)_k = \int_\Omega \mu v_1 v_2 dx \quad \forall v_1, v_2 \in V_k, \tag{5.4.4}$$

and $Q_{a,k} : V_k \to \tilde{V}_k$, where

$$\tilde{V}_k = \{v \in V_k : (\mu v, 1) = 0\}$$

or

$$\tilde{V}_k = \{v \in V_k : (v, 1) = 0\},$$

is the orthogonal projection with respect to the inner product $(\cdot,\cdot)_k$, i.e.,

$$(Q_{a,k}v_1, v_2)_k = (v_1, v_2)_k \quad \forall v_1 \in V_k, v_2 \in \tilde{V}_k. \tag{5.4.5}$$

For the convergence analysis, we also define the Ritz projection operators $P_k : H^1(\Omega) \to \tilde{V}_k$ and $P_{k,\mu} : H^1(\Omega) \to V_k$. If $\alpha = 0$ in (5.1.5), then $P_k : H^1(\Omega) \to \tilde{V}_k$ is defined by

$$(\epsilon^{-1}\nabla \times (\zeta - P_k\zeta), \nabla \times v) = 0 \quad \forall \zeta \in H^1(\Omega), v \in \tilde{V}_k. \tag{5.4.6}$$

If $\alpha \neq 0$ in (5.1.5), then $P_{k,\mu} : H^1(\Omega) \to V_k$ is defined by

$$(\epsilon^{-1}\nabla \times (\zeta - P_k\zeta), \nabla \times v) + \alpha(\mu(\zeta - P_k\zeta), v) = 0 \quad \forall \zeta \in H^1(\Omega), v \in V_k. \tag{5.4.7}$$

The following is the standard two-sided symmetric k^{th} level multigrid iteration scheme. For $p = 1$ it is the V-cycle algorithm with m presmoothing steps and m postsmoothing steps, and for $p = 2$ it is the W-cycle algorithm with m presmoothing steps and m postsmoothing steps.

5.4.1 The k^{th} Level Iteration

The k^{th} level iteration with initial guess z_0 yields $MG(k, z_0, g)$ as an approximate solution to the equation

$$\begin{cases} A_k z &= g, \\ (\mu z, 1) &= 0, \end{cases} \tag{5.4.8}$$

or

$$A_k z + \alpha M_k z = g \tag{5.4.9}$$

when $\alpha \neq 0$. For $k = 1$, $MG(1, z_0, g)$ is the solution obtained from an exact solver.

For $k > 1$, $MG(k, z_0, g)$ is obtained recursively in three steps.

Presmoothing Step. For $1 \le l \le m$,

$$\begin{cases} \tilde{z}_l = z_{l-1} + \dfrac{1}{\Lambda_k}(g - A_k z_{l-1}), \\ z_l = Q_{\mu,k}\tilde{z}_l, \end{cases} \tag{5.4.10}$$

or

$$z_l = z_{l-1} + \frac{1}{\Lambda_k}(g - \alpha M_k z_{l-1} - A_k z_{l-1}) \tag{5.4.11}$$

when $\alpha \neq 0$, where m is a positive integer independent of k, and Λ_k dominates the spectral radius of $A_k + \alpha M_k$.

Correction Step. Let $\bar{g} = I_k^{k-1}(g - A_k z_m)$ or $\bar{g} = I_k^{k-1}(g - \alpha M_k z_m - A_k z_m)$ and $q_i \in V_{k-1}(0 \leq i \leq p, p = 1 \text{ or } 2)$ be defined recursively by

$$q_0 = 0 \tag{5.4.12}$$

and

$$q_i = MG(k-1, q_{i-1}, \bar{g}) \quad \text{for} \quad 1 \leq i \leq p. \tag{5.4.13}$$

Then

$$z_{m+1} = z_m + I_{k-1}^k q_p. \tag{5.4.14}$$

Postsmoothing Step. For $m + 2 \leq l \leq 2m + 1$, let

$$\begin{cases} \tilde{z}_l = z_{l-1} + \dfrac{1}{\Lambda_k}(g - A_k z_{l-1}), \\ z_l = Q_{a,k}\tilde{z}_l, \end{cases} \tag{5.4.15}$$

or

$$z_l = z_{l-1} + \frac{1}{\Lambda_k}(g - \alpha M_k z_{l-1} - A_k z_{l-1}) \tag{5.4.16}$$

if $\alpha \neq 0$. Then the final output of the k^{th} level iteration is

$$MG(k, z_0, g) = z_{2m+1}. \tag{5.4.17}$$

5.4.2 The Full Multigrid Algorithms

We use a nested iteration to compute $\kappa_{l,k}^{\xi}$ and $w_k \in V_k$ so that $\kappa_{k,l}^{\xi}$ approximates the stress intensity factor κ_l^{ξ}, and $\xi_k = \sum_{0<\sigma_l<1} \kappa_{l,k}^{\xi} s_l + w_k$ approximates the solution ξ of (5.1.5). Then we approximate ξ in (5.1.3) by ξ_N (the approximation of ξ on the finest level), and use a nested iteration to compute $\kappa_{l,k}^{\phi}$ and $v_k \in V_k$ so that

$\phi_k = \sum_{0<\sigma_l<1} \kappa_{l,k}^{\phi} s + v_k$ approximates ϕ. The full multigrid algorithm is described as follows.

Algorithm 5.4.1. (Full Multigrid Algorithm for ξ.) *Let $Q_{a,k} = Q_{\mu,k}$ in (5.4.10) and (5.4.15) when $\alpha \neq 0$. For $k = 1$, ξ_1 is the exact solution of (5.4.8), where $g_1 \in V_1$ is defined by*

$$(g_1, v)_1 = (\boldsymbol{f}, \epsilon^{-1}\nabla \times v) \quad \forall v \in V_1,$$

and we set $w_1 = \xi_1$.

For $2 \leq k \leq N$, the stress intensity factors $\kappa_{l,k}^{\xi}$ are computed by the following extraction formula:

$$\begin{aligned}\kappa_{l,k}^{\xi} = \frac{1}{2\sigma_l}\{ &\int_{\Omega} (\epsilon^{-1}(\nabla \times \boldsymbol{f}) s_l^* - \alpha\mu\xi_{k-1} s_l^* + \epsilon^{-1}\xi_{k-1}\Delta s_l^*) dx \\ &- \int_{\partial\Omega} \epsilon^{-1}(\boldsymbol{n} \times \boldsymbol{f}) s_l^* ds - \int_{\Gamma} [\epsilon^{-1}\boldsymbol{n} \times \boldsymbol{f}] s_l^* ds\},\end{aligned} \tag{5.4.18}$$

and $w_k \in V_k$ is obtained recursively by

$$\begin{cases} w_{k,0} &= w_{k-1}, \\ w_{k,l} &= MG(k, w_{k,l-1}, g_k) \quad \text{for} \quad 1 \leq l \leq n, \\ w_k &= w_{k,n}, \end{cases} \tag{5.4.19}$$

where n is a positive integer independent of k, and $g_k \in V_k$ is defined by

$$(g_k, v)_k = \int_{\Omega} \{\boldsymbol{f} \cdot (\epsilon^{-1}\nabla \times v) - \epsilon^{-1} \sum_{0<\sigma_l<1} \kappa_{l,k}^{\xi} \nabla s_l \cdot \nabla v - \alpha\mu v \sum_{0<\sigma_l<1} \kappa_{l,k}^{\xi} s_l\} dx \quad \forall v \in V_k. \tag{5.4.20}$$

Then we define

$$\xi_k = w_k + \sum_{0<\sigma_l<1} \kappa_{l,k}^{\xi} s_l. \tag{5.4.21}$$

Similarly, we define a full multigrid algorithm for ϕ.

Algorithm 5.4.2. (Full Multigrid Algorithm for ϕ.) *Let $Q_{a,k} = Q_{1,k}$ in (5.4.10) and (5.4.15). For $k = 1$, ϕ_1 is the exact solution of (5.4.8), where $g_1 \in V_1$ is defined by*

$$(g_1, v)_1 = (\mu \xi_N, v) \quad \forall v \in V_1,$$

and we set $v_1 = \phi_1$.

For $2 \le k \le N$, the stress intensity factors $\kappa^{\phi}_{l,k}$ are computed by the extraction formula:

$$\kappa^{\phi}_{l,k} = \frac{1}{2\sigma_l} \int_{\Omega} (\mu \xi_N s_l^* + \epsilon^{-1} \phi_{k-1} \Delta s_l^*) dx \tag{5.4.22}$$

and $v_k \in V_k$ is obtained recursively by

$$\begin{cases} v_{k,0} &= v_{k-1}, \\ v_{k,l} &= MG(k, v_{k,l-1}, g_k) \quad \textit{for} \quad 1 \le l \le n, \\ v_k &= v_{k,n}, \end{cases} \tag{5.4.23}$$

where $g_k \in V_k$ is defined by

$$(g_k, v)_k = \int_{\Omega} (\mu \xi_N v - \epsilon^{-1} \sum_{0<\sigma_l<1} \kappa^{\phi}_{l,k} \nabla s_l \cdot \nabla v) dx \quad \forall v \in V_k. \tag{5.4.24}$$

Then we define

$$\phi_k = v_k + \sum_{0<\sigma_l<1} \kappa^{\phi}_{l,k} s_l \tag{5.4.25}$$

and the approximation of $\mathring{\boldsymbol{u}}$ by

$$\mathring{\boldsymbol{u}}_h = \epsilon^{-1} \boldsymbol{\nabla} \times v_k + \sum_{0<\sigma_l<1} \kappa^{\phi}_{l,k} \epsilon^{-1} \boldsymbol{\nabla} \times s_l.$$

5.5 Convergence Analysis

In this section we prove the convergence of Algorithm 5.4.1 and Algorithm 5.4.2 for the case when $\alpha \ge 0$.

In order to avoid the proliferation of constants, we will use the notation $A \lesssim B$ to represent the statement that A is bounded by B multiplied by a constant which is independent of the variables, the mesh sizes and the grid levels. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$, and the notation $A \lesssim_\epsilon B$ indicates that the constant may depend on ϵ.

First we derive a convergence result for the k^{th} level symmetric W-cycle multigrid algorithm applied to our interface problems. Recall that $MG(k, z_0, g)$ is the approximate solution of (5.4.8) obtained by the k^{th} level iteration scheme with initial guess z_0.

Lemma 5.5.1. *Let $p = 2$ in the k^{th} level iteration scheme. Let $\delta \in (0, 1)$, $\epsilon \in (0, \sigma_1)$ and $\sigma_\epsilon = 1 - \sigma_1 + \epsilon$, we have*

$$\|z - MG(k, z_0, g)\|_{H^1(\Omega)} \leq \delta \|z - z_0\|_{H^1(\Omega)} \quad \forall k \geq 1, \tag{5.5.1}$$

and

$$\|z - MG(k, z_0, g)\|_{H^{\sigma_\epsilon}(\Omega)} \leq \delta \|z - z_0\|_{H^{\sigma_\epsilon}(\Omega)} \quad \forall k \geq 1, \tag{5.5.2}$$

provided that the number of smoothing steps m is sufficiently large.

Proof. The proof of (5.5.1) is essentially identical to the proof of Theorem 1 in the paper [4]. The proof of (5.5.2) below is similar to the proof of Lemma 3.2 in [22], where the boundary condition is different.

We follow the methodology in [22]. For simplicity, we assume that $\alpha = 1$ in (5.1.5). Let $\tilde{A}_k = A_k + M_k$. First we consider the two-grid algorithm where q_p in (5.4.14) is replaced by

$$q = (\tilde{A}_{k-1})^{-1} \bar{g} = P_{k-1}(z - z_m), \tag{5.5.3}$$

and there is no postsmoothing steps.

Let $R_k : V_k \longrightarrow V_k$ be defined by

$$R_k = I - \frac{1}{\Lambda_k} \tilde{A}_k. \tag{5.5.4}$$

Then, by (5.5.4) and (5.4.11), we have

$$z - z_m = R_k^m (z - z_0). \tag{5.5.5}$$

It follows from (5.5.3) and (5.5.5) that the error of the output $\tilde{z} = z_m + q$ of the two-grid method is given by

$$z - \tilde{z} = z - z_m - q = (I - P_{k-1})(z - z_m) = (I - P_{k-1}) R_k^m (z - z_0). \tag{5.5.6}$$

For $0 \le s \le 1$, we define

$$|||v|||_s = \sqrt{(\tilde{A}_k^s v, v)_k} \quad \forall v \in V_k. \tag{5.5.7}$$

By the spectral decomposition for positive definite operators, (5.5.7) and a slight modification of [20, Lemma 6.2.8], we have

$$|||v|||_1 \le [\rho(\tilde{A}_k^{1-s})]^{1/2} \, |||v|||_s \lesssim h_k^{s-1} \, |||v|||_s \quad \forall v \in V_k, 0 \le s \le 1, \tag{5.5.8}$$

and since $\Lambda_k = C h_k^{-2}$ dominates the spectral radius of $\tilde{A}_k$,

$$|||R_k v|||_s \le |||v|||_s \quad \forall v \in V_k, 0 \le s \le 1. \tag{5.5.9}$$

The effect of the smoothing step is measured by the following smoothing property:

$$\begin{aligned} |||R_k^m v|||_1 &= ((\tilde{A}_k R_k^m v, R_k^m v)_k)^{1/2} = ((\tilde{A}_k R_k^{2m} v, v)_k)^{1/2} \\ &\le \Lambda_k^{s/2} \left[\rho(\Lambda_k^{-s} \tilde{A}_k^s R_k^{2m}) \right]^{1/2} (\tilde{A}_k^{1-s} v, v)_k^{1/2} \quad \text{(by spectral decomposition)} \\ &\lesssim h_k^{-s} [\sup_{0 \le t \le 1} t^s (1-t)^{2m}]^{1/2} \, |||v|||_{1-s} \\ &\lesssim h_k^{-s} m^{-s/2} \, |||v|||_{1-s} \quad \forall v \in V_k, 0 \le s \le 1. \end{aligned} \tag{5.5.10}$$

The effect of the correction step is given by the following approximation property:

$$\|v - P_{k-1} v\|_{H^{\sigma_\epsilon}(\Omega)} \lesssim_\epsilon h_k^{\sigma_1 - \epsilon} |v - P_{k-1} v|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega). \tag{5.5.11}$$

We will establish (5.5.11) by a duality argument. Let $L \in [H^{\sigma_\epsilon}(\Omega)]'$ and $\zeta \in H^1(\Omega)$ satisfy

$$(\epsilon^{-1}\boldsymbol{\nabla} \times \zeta, \boldsymbol{\nabla} \times v) + (\mu\zeta, v) = L(v) \quad \forall v \in H^1(\Omega). \tag{5.5.12}$$

It is well known (cf. [53]) that $\zeta \in H^{1+\sigma_1-\epsilon}(\Omega)$ and

$$\|\zeta\|_{H^{1+\sigma_1-\epsilon}(\Omega)} \lesssim_\epsilon \|L\|_{[H^{\sigma_\epsilon}(\Omega)]'}. \tag{5.5.13}$$

Let Π_k be the nodal interpolation operator associated with V_k. It follows from standard interpolation error estimates (cf. Section 2.4) that

$$\|\zeta - \Pi_k\zeta\|_{H^1(\Omega)} \lesssim_\epsilon h_k^{\sigma_1-\epsilon}\|L\|_{[H^{\sigma_\epsilon}(\Omega)]'}. \tag{5.5.14}$$

We can modify the proof for Friedrichs' inequality (cf. [20, Lemma 4.3.14]) and have

$$\|u - \tilde{u}\|_{H^1(\Omega)} \lesssim |u|_{H^1(\Omega)}, \tag{5.5.15}$$

where $u \in H^1(\Omega)$ and $\tilde{u} = \frac{1}{|\Omega|}\int_\Omega \mu u\, dx$.

Using (5.4.7), (5.5.12), (5.5.15) and (5.5.14), we have

$$\begin{aligned}
L(v - P_{k-1}v) &= (\epsilon^{-1}\boldsymbol{\nabla} \times \zeta, \boldsymbol{\nabla} \times (v - P_{k-1}v)) + (\mu\zeta, v - P_{k-1}v) \\
&= (\epsilon^{-1}\boldsymbol{\nabla} \times (\zeta - \Pi_k\zeta), \boldsymbol{\nabla} \times (v - P_{k-1}v)) + (\mu(\zeta - \Pi_k\zeta), v - P_{k-1}v) \\
&\lesssim \|\zeta - \Pi_k\zeta\|_{H^1(\Omega)}\|v - P_{k-1}v\|_{H^1(\Omega)} \\
&\lesssim h_k^{\sigma_1-\epsilon}\|L\|_{[H^{\sigma_\epsilon}(\Omega)]'}|v - P_{k-1}v|_{H^1(\Omega)}.
\end{aligned} \tag{5.5.16}$$

The estimate (5.5.11) follows from (5.5.16) and the duality formula

$$\|\eta\|_{H^{\sigma_\epsilon}(\Omega)} = \sup[L(\eta)/\|L\|_{[H^{\sigma_\epsilon}(\Omega)]'}] \quad \forall \eta \in H^{\sigma_\epsilon}(\Omega), \tag{5.5.17}$$

where the supremum is taken over all $L \in [H^{\sigma_\epsilon}(\Omega)]' \setminus \{0\}$.

The final ingredient is the relation between the mesh dependent norm $|||\cdot|||_s$ and the Sobolev norm $\|\cdot\|_{H^s(\Omega)}$ on V_k. First of all, we have $|v|_{H^1(\Omega)} \approx |||v|||_1$ and $\|v\|_{L_2(\Omega)} \lesssim |||v|||_0$ for all $v \in V_k$. Interpolating these estimates (cf. [20, Proposition 14.1.5]), we have

$$\|v\|_{H^s(\Omega)} \lesssim |||v|||_s \quad \forall v \in V_k, 0 \leq s \leq 1. \tag{5.5.18}$$

Meanwhile, there exists (cf. [26, 59]) an interpolation operator $\pi_k : L_2(\Omega) \longrightarrow V_k$ such that

$$|||\pi_k v|||_0 \lesssim \|\pi_k v\|_{L_2(\Omega)} \lesssim \|v\|_{L_2(\Omega)} \quad \forall v \in L_2(\Omega), \tag{5.5.19}$$

$$|||\pi_k v|||_1 \lesssim \|\pi_k v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega), \tag{5.5.20}$$

$$\pi_k v = v \quad \forall v \in V_k. \tag{5.5.21}$$

For $0 < s < 1$, we can interpolate (5.5.19) and (5.5.20) (cf. [20, Proposition 14.1.5, Theorem 14.2.3]) to obtain

$$|||\pi_k v|||_s \lesssim \|v\|_{H^s(\Omega)} \quad \forall v \in H^s(\Omega), 0 \leq s \leq 1. \tag{5.5.22}$$

Combining (5.5.21) and (5.5.22), we find

$$|||v|||_s \lesssim \|v\|_{H^s(\Omega)} \quad v \in V_k, 0 \leq s \leq 1. \tag{5.5.23}$$

Therefore, from (5.5.6),(5.5.10), (5.5.11), (5.5.18) and (5.5.23), we have the following error estimate for the two-grid algorithm:

$$\begin{aligned} \|z - \tilde{z}\|_{H^{\sigma_\epsilon}(\Omega)} &\lesssim h_k^{\sigma_1 - \epsilon} |R_k^m (z - z_m)|_{H^1(\Omega)} \\ &\lesssim m^{[-\sigma_1+\epsilon]/2} \|z - z_0\|_{H^{\sigma_\epsilon}(\Omega)}. \end{aligned} \tag{5.5.24}$$

Now we estimate the error for the one-sided W-cycle k^{th} level iteration. Let $\gamma_m = m^{[-\sigma_1+\epsilon]/2}$ and suppose that the error of the $(k-1)^{st}$ level iteration in the

$H^{\sigma_\epsilon}(\Omega)$ norm is reduced by a factor η. Then it follows from (5.4.14), (5.5.6) and (5.5.24) that

$$
\begin{aligned}
\|z - MG(k, z_0, g)\|_{H^{\sigma_\epsilon}(\Omega)} &\le \|z - \tilde{z}\|_{H^{\sigma_\epsilon}(\Omega)} + \|q - q_2\|_{H^{\sigma_\epsilon}(\Omega)} \\
&\le C_\epsilon \gamma_m \|z - z_0\|_{H^{\sigma_\epsilon}(\Omega)} + \eta^2 \|q\|_{H^{\sigma}(\Omega)}. \qquad (5.5.25)
\end{aligned}
$$

From (5.5.8), (5.5.11), (5.5.18) and (5.5.23), we have

$$
\begin{aligned}
\|P_{k-1} v\|_{H^{\sigma_\epsilon}(\Omega)} &\le \|v - P_{k-1} v\|_{H^{\sigma_\epsilon}(\Omega)} + \|v\|_{H^{\sigma_\epsilon}(\Omega)} \\
&\lesssim_\epsilon h_k^{\sigma_1 - \epsilon} |v|_{H^1(\Omega)} + \|v\|_{H^{\sigma_\epsilon}(\Omega)} \\
&\lesssim_\epsilon \|v\|_{H^{\sigma_\epsilon}(\Omega)} \quad \forall v \in V_k. \qquad (5.5.26)
\end{aligned}
$$

Combining (5.5.5), (5.5.9), (5.5.18), (5.5.22) and (5.5.26), we obtain

$$
\|q\|_{H^{\sigma_\epsilon}(\Omega)} = \|P_{k-1}(z - z_m)\|_{H^{\sigma_\epsilon}(\Omega)} \le C_\epsilon' \|z - z_0\|_{H^{\sigma_\epsilon}(\Omega)}. \qquad (5.5.27)
$$

The estimates (5.5.23), (5.5.25) and (5.5.27) together imply that

$$
\|z - MG(k, z_0, g)\|_{H^{\sigma_\epsilon}(\Omega)} \le (C_\epsilon \gamma_m + C_\epsilon' \eta^2) \|z - z_0\|_{H^{\sigma_\epsilon}(\Omega)}. \qquad (5.5.28)
$$

For m sufficiently large, we have $\gamma_m < (4 C_\epsilon C_\epsilon')^{-1}$ and

$$
\eta_m = \left[1 - (1 - 4 C_\epsilon C_\epsilon' \gamma_m)^{1/2}\right] / (2 C_\epsilon') \qquad (5.5.29)
$$

is a fixed point of the map $T(\eta) = C_\epsilon \gamma_m + C_\epsilon' \eta^2$. Since the first level iteration is an exact solver, it follows from (5.5.29) and mathematical induction that

$$
\|z - MG(k, z_0, g)\|_{H^{\sigma_\epsilon}(\Omega)} \le \eta_m \|z - z_0\|_{H^{\sigma_\epsilon}(\Omega)} \quad \text{for} \quad k \ge 1. \qquad (5.5.30)
$$

As $\lim_{m\to\infty} \eta_m = 0$, the estimate (5.5.2) follows from (5.5.30). □

Remark 5.5.2. For the symmetric W-cycle k^{th} level iteration, we have

$$
z - MG(k, z_0, g) = R_k^m (z - z_{m+1}). \qquad (5.5.31)
$$

Combining (5.5.9) and (5.5.31), we can prove (5.5.2) for the symmetric W-cycle k^{th} level iteration.

With the help of Lemma 5.5.1, we can prove the following result by following the methodology of [23].

Theorem 5.5.3. *Let $p = 2$ in the k^{th} level iteration scheme, $\epsilon \in (0, \sigma_1)$, $\sigma_\epsilon = 1 - \sigma_1 + \epsilon$, and the number of smoothing steps m be sufficiently large so that the k^{th} level iteration scheme is a contraction. If the number of nested iterations n is sufficiently large, then we have*

$$\sum_{0<\sigma_l<1} |\kappa_l^\xi - \kappa_{l,k}^\xi| \leq C_\epsilon h_k^{1+\sigma_1-\epsilon} \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1 \tag{5.5.32}$$

and

$$\|w_\xi - w_k\|_{H^{\sigma_\epsilon}(\Omega)} \leq C_\epsilon h_k^{1+\sigma_1-\epsilon} \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1. \tag{5.5.33}$$

Moreover we have the following estimates in the H^1 norm:

$$\|w_\xi - w_k\|_{H^1(\Omega)} \leq C h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1. \tag{5.5.34}$$

Let

$$\xi_k = \sum_{0<\sigma_l<1} \kappa_{l,k}^\xi s_l + w_k, \tag{5.5.35}$$

then (5.5.33) and (5.5.34) lead to the following estimate:

$$\|\xi - \xi_k\|_{H^1(\Omega)} \leq C h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1. \tag{5.5.36}$$

Proof. We will establish (5.5.32) and (5.5.33) through recursive estimates. By (5.3.7) and (5.4.18) along with (5.3.3) and (5.4.21), we know that, for $1 \leq l \leq L$,

$$\begin{aligned} |\kappa_l^\xi - \kappa_{l,k}^\xi| &= \frac{1}{2\sigma_1} \left| \int_\Omega \epsilon^{-1} (\xi - \xi_{k-1}) \Delta s_l^* dx \right| \\ &\lesssim \|\xi - \xi_{k-1}\|_{L^2(\Omega)} \|\Delta s_l^*\|_{L^2(\Omega)} \end{aligned} \tag{5.5.37}$$

$$\lesssim \|\hat{w}_{k-1} - w_{k-1}\|_{H^{\sigma_\epsilon}(\Omega)},$$

where

$$\hat{w}_k = \xi - \sum_{0<\sigma_l<1} \kappa_{l,k}^{\xi} s_l = \sum_{0<\sigma_l<1} (\kappa_l^{\xi} - \kappa_{l,k}^{\xi}) s_l + w_\xi = (\xi - \xi_k) + w_k. \tag{5.5.38}$$

Let $a_k = \sum\limits_{0<\sigma_l<1} |\kappa_l^{\xi} - \kappa_{l,k}^{\xi}|$, and $b_k = \|\hat{w}_k - w_k\|_{H^{\sigma_\epsilon}(\Omega)}$. So (5.5.37) says that

$$a_k \lesssim b_{k-1}. \tag{5.5.39}$$

To estimate b_k, we begin with

$$b_k \leq \|\hat{w}_k - P_k\hat{w}_k\|_{H^{\sigma_\epsilon}(\Omega)} + \|P_k\hat{w}_k - w_k\|_{H^{\sigma_\epsilon}(\Omega)}. \tag{5.5.40}$$

For the first term on the right-hand side of equation (5.5.40), by using the elliptic regularity estimate (5.3.5) and standard finite element tools (cf. Section 2.4), we have the following estimate

$$\|\hat{w}_k - P_k\hat{w}_k\|_{H^{\sigma_\epsilon}(\Omega)} \leq \|w_\xi - P_k w_\xi\|_{H^{\sigma_\epsilon}(\Omega)} + \sum_{0<\sigma_l<1} |\kappa_l^{\xi} - \kappa_{l,k}^{\xi}| \|s_l - P_k s_l\|_{H^{\sigma_\epsilon}(\Omega)} \tag{5.5.41}$$

$$\lesssim_\epsilon h_k^{1+\sigma_1-\epsilon} \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} + h_k^{2(\sigma_1-\epsilon)} a_k.$$

The second term on the right-hand side of (5.5.40) can be estimated as follows:

$$\|P_k\hat{w}_k - w_k\|_{H^{\sigma_\epsilon}(\Omega)} \leq \delta^n \|P_k\hat{w}_k - w_{k-1}\|_{H^{\sigma_\epsilon}(\Omega)}, \tag{5.5.42}$$

by Lemma 5.5.1, since $P_k\hat{w}$ is the exact solution of (5.4.9) whose right-hand side is given by (5.4.19) and w_k is its approximate solution.

Now we estimate $\|P_k\hat{w}_k - w_{k-1}\|_{H^{\sigma_\epsilon}(\Omega)}$ as follows:

$$\|P_k\hat{w}_k - w_{k-1}\|_{H^{\sigma_\epsilon}(\Omega)} \leq \|P_k\hat{w}_k - \hat{w}_k\|_{H^{\sigma_\epsilon}(\Omega)} + \|\hat{w}_k - w_\xi\|_{H^{\sigma_\epsilon}(\Omega)}$$
$$+ \|w_\xi - \hat{w}_{k-1}\|_{H^{\sigma_\epsilon}(\Omega)} + \|\hat{w}_{k-1} - w_{k-1}\|_{H^{\sigma_\epsilon}(\Omega)}$$

$$\lesssim_\epsilon h_k^{1+\sigma_1-\epsilon}\|\nabla\times\boldsymbol{f}\|_{L_2(\Omega)} + a_k + a_{k-1} + b_{k-1}, \tag{5.5.43}$$

by (5.5.38), (5.5.41) and the definitions of a_k and b_k.

Combining (5.5.39)-(5.5.43) we find, for $k \geq 2$,

$$\begin{aligned} b_k &\lesssim_\epsilon (h_k^{1+\sigma_1-\epsilon}\|\nabla\times\boldsymbol{f}\|_{L_2(\Omega)} + h_k^{2(\sigma_1-\epsilon)}a_k) + \delta^n(h_k^{1+\sigma_1-\epsilon}\|\nabla\times\boldsymbol{f}\|_{L_2(\Omega)} \\ &\quad + a_k + a_{k-1} + b_{k-1}) \\ &\leq C_\epsilon(h_k^{2(\sigma_1-\epsilon)} + \delta^n)b_{k-1} + C_\epsilon\delta^n b_{k-2} + C_\epsilon h_k^{1+\sigma_1-\epsilon}\|\nabla\times\boldsymbol{f}\|_{L_2(\Omega)}, \end{aligned} \tag{5.5.44}$$

where C_ϵ is a positive constant.

Therefore (5.5.44) leads to the estimate

$$b_k \leq \beta b_{k-1} + \beta b_{k-2} + C_* h_k^{1+\sigma_1-\epsilon}, \tag{5.5.45}$$

provided that $C_\epsilon h_k^{2(\sigma_1-\epsilon)} + C_\epsilon\delta^n < \beta$ for some positive constant β and

$$C_* = C_\epsilon\|\nabla\times\boldsymbol{f}\|_{L_2(\Omega)}. \tag{5.5.46}$$

Later we will identify the choice of β.

We reformulate (5.5.45) as

$$\begin{bmatrix} b_{k-1} \\ b_k \end{bmatrix} \leq \begin{bmatrix} 0 & 1 \\ \beta & \beta \end{bmatrix} \begin{bmatrix} b_{k-2} \\ b_{k-1} \end{bmatrix} + C_* h_k^{1+\sigma_1-\epsilon} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{5.5.47}$$

where the vector inequality is interpreted component-wise.

Let $\boldsymbol{M} = \begin{bmatrix} 0 & 1 \\ \beta & \beta \end{bmatrix}$ and rewrite (5.5.47) as

$$\begin{bmatrix} b_{k-1} \\ b_k \end{bmatrix} \leq \boldsymbol{M} \begin{bmatrix} b_{k-2} \\ b_{k-1} \end{bmatrix} + C_* h_k^{1+\sigma_1-\epsilon} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{5.5.48}$$

For any given $\epsilon \in (0, \sigma_1)$ and $\beta > 0$, there exist sufficiently large k^* and n^* depending on β such that $C_\epsilon h_k^{2(\sigma_1-\epsilon)} + C_\epsilon\delta^n < \beta$ for $k \geq k^*$ and $n \geq n^*$. So for $k \geq k^*$ and $n \geq n^*$, (5.5.48) is valid.

By iterating (5.5.48), we obtain

$$\begin{bmatrix} b_{k-1} \\ b_k \end{bmatrix} \le \boldsymbol{M}^{k-k^*} \begin{bmatrix} b_{k^*-1} \\ b_{k^*} \end{bmatrix} + C_*(h_k^{1+\sigma_1-\epsilon} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + h_{k-1}^{1+\sigma_1-\epsilon} \boldsymbol{M} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.5.49}$$
$$+ \ldots + h_{k^*+1}^{1+\sigma_1-\epsilon} \boldsymbol{M}^{k-k^*-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}).$$

By a direct computation, we have

$$\boldsymbol{M}^2 = \beta(I + \boldsymbol{M}), \tag{5.5.50}$$

where $\boldsymbol{I}$ is the 2×2 identity matrix.

By (5.5.50), we have

$$\|\boldsymbol{M}^2\|_\infty \le 2\beta$$

and hence

$$\|\boldsymbol{M}^{2\iota}\|_\infty \le (2\beta)^\iota$$

and

$$\|\boldsymbol{M}^{2\iota+1}\|_\infty \le (2\beta)^\iota,$$

where ι is a positive integer. So for sufficiently small β, i.e., for k and n sufficiently large, we have

$$\|\boldsymbol{M}^t\|_\infty \le 2^{-(1+\sigma_1)t} \quad \text{for} \quad t = 2, ... \tag{5.5.51}$$

Then (5.5.49) and (5.5.51) implies that, for $k \ge k^* + 1$ and $n \ge n^*$,

$$\begin{aligned} b_k \le & 2^{-(1+\sigma_1)(k-k^*)} b_{k^*} + C_* h_k^{1+\sigma_1-\epsilon}(1 + 2^{-\epsilon} + \ldots \\ & + 2^{-(k-k^*-1)\epsilon}) \|\boldsymbol{\nabla} \times \boldsymbol{f}\|_{L_2(\Omega)}, \\ \le & 2^{-(1+\sigma_1)(k-k^*)} b_{k^*} + \frac{C_*}{1-2^{-\epsilon}} h_k^{1+\sigma_1-\epsilon} \|\boldsymbol{\nabla} \times \boldsymbol{f}\|_{L_2(\Omega)}. \end{aligned} \tag{5.5.52}$$

On the other hand, we have

$$a_1 = \sum_{0<\sigma_l<1} |\kappa_l| \lesssim \|\nabla \times \boldsymbol{f}\|_{L^2(\Omega)}, \tag{5.5.53}$$

$$b_1 = \|\xi - P_1\xi\|_{H^{\sigma_\epsilon}(\Omega)} \lesssim \|\xi\|_{H^1(\Omega)} \lesssim \|\nabla \times \boldsymbol{f}\|_{L^2(\Omega)}, \tag{5.5.54}$$

Combining (5.5.44),(5.5.53) and (5.5.54), we obtain, for $1 \le k \le k^*$,

$$b_k \lesssim_\epsilon \|\nabla \times \boldsymbol{f}\|_{L^2(\Omega)} \lesssim_\epsilon h_k^{1+\sigma_1-\epsilon}\|\nabla \times \boldsymbol{f}\|. \tag{5.5.55}$$

Combining (5.5.52) and (5.5.55), we conclude that

$$b_k \lesssim_\epsilon h_k^{1+\sigma_1-\epsilon}\|\nabla \times \boldsymbol{f}\|_{L^2(\Omega)} \quad \text{for} \quad k \ge 1. \tag{5.5.56}$$

We have established (5.5.33), and (5.5.32) follows directly from (5.5.39).

Now we consider the error estimate (5.5.34). First note that if we let ϵ be $\frac{\sigma_1}{2}$ in (5.5.32) , then we obtain

$$a_k \le C h_k^{1+\sigma_1/2}\|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)}, \tag{5.5.57}$$

where the constant C no longer depends on ϵ.

Let us denote

$$\bar{b}_k := \|\hat{w}_k - w_k\|_{H^1(\Omega)}. \tag{5.5.58}$$

We have the following estimate

$$\|w_\xi - \hat{w}_k\|_{H^{1-\sigma_1+\epsilon}(\Omega)} \lesssim \|w_\xi - \hat{w}_k\|_{H^1(\Omega)} \lesssim a_k \tag{5.5.59}$$

by (5.5.38).

Also, by the elliptic regularity estimate assumption, we have the estimate

$$\bar{b}_1 = \|\hat{w}_1 - w_1\|_{H^1(\Omega)} = \|\xi - P_1\xi\|_{H^1(\Omega)} \lesssim \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)}. \tag{5.5.60}$$

We may also obtain the analogue of (5.5.40)-(5.5.43). From (5.5.58), we have

$$\bar{b}_k \le \|\hat{w}_k - P_k\hat{w}_k\|_{H^1(\Omega)} + \|P_k\hat{w}_k - w_k\|_{H^1(\Omega)}. \tag{5.5.61}$$

By (5.5.38), (5.5.57) and the analogue of (5.5.41), we have

$$\|\hat{w}_k - P_k\hat{w}_k\|_{H^1(\Omega)} \lesssim h_k\|\nabla \times \boldsymbol{f}\|_{L^2(\Omega)} + h_k^{\sigma_1-\epsilon}a_k \tag{5.5.62}$$

$$\lesssim h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)}.$$

By Lemma 5.5.1, we have

$$\|P_k \hat{w}_k - w_k\|_{H^1(\Omega)} \leq \delta^n \|P_k \hat{w}_k - w_{k-1}\|_{H^1(\Omega)}. \tag{5.5.63}$$

By the analogue of (5.5.44), we have

$$\|P_k \hat{w}_k - w_{k-1}\|_{H^1(\Omega)} \lesssim h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} + a_k + a_{k-1} + \bar{b}_{k-1} \lesssim h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} + \bar{b}_{k-1}. \tag{5.5.64}$$

So (5.5.61)-(5.5.64) imply that

$$\bar{b}_k \lesssim \delta^n \bar{b}_{k-1} + h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \tag{5.5.65}$$

or

$$\bar{b}_k \leq C_\dagger (\delta^n \bar{b}_{k-1} + h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)}) \tag{5.5.66}$$

for some constant $C_\dagger$. For n sufficiently large so that $C_\dagger \delta^n < 1/4$, we can iterate (5.5.66) and apply (5.5.60) to get the estimate

$$\begin{aligned} \bar{b}_k &\leq (C_\dagger \delta^n)^{k-1} \bar{b}_1 + C_\dagger \|\boldsymbol{\nabla} \times \boldsymbol{f}\|_{L_2(\Omega)} \sum_{i=2}^{k} (C_\dagger \delta^n)^{k-i} h_i \\ &\lesssim h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \text{for} \quad k \geq 1, \end{aligned} \tag{5.5.67}$$

which is (5.5.34). □

Remark 5.5.4. Because of (5.3.3), (5.5.35), (5.5.32), and (5.5.33), we also have the estimate

$$\|\xi - \xi_k\|_{L_2(\Omega)} \leq C h_k^{1+\sigma_1-\epsilon} \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1. \tag{5.5.68}$$

Remark 5.5.5. For the case $\alpha < 0$, it is a symmetric indefinite problem and there are several multigrid schemes developed [3, 47, 48, 9, 63, 8] in the literature. We will not discuss this case here.

We use $\xi_N = \sum_{0<\sigma_l<1} \kappa_{l,N}^{\xi} s_l + v_N$, where $\mathscr{T}_N$ is the finest triangulation, as the approximation to the solution ξ of (5.1.5) in (5.1.3). Applying Algorithm 5.4.2 to the equation (5.1.3), we obtain the approximation $\kappa_{l,k}^{\phi}$ and v_k, for $1 \leq k \leq N$..

Theorem 5.5.6. *Let $p = 2$ in the k^{th} level iteration scheme, $\epsilon \in (0, \sigma_1)$, $\sigma_\epsilon = 1 - \sigma_1 + \epsilon$, and the number of smoothing steps m be sufficiently large so that kth level iteration scheme is a contraction scheme. If the number of nested steps n is sufficiently large, then we have*

$$\sum_{0<\sigma_l<1} |\kappa_l^{\phi} - \kappa_{l,k}^{\phi}| \leq C h_k^{1+\sigma_1-\epsilon} \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1, \tag{5.5.69}$$

$$\|w_\phi - v_k\|_{H^1(\Omega)} \leq C h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1. \tag{5.5.70}$$

Let $\phi_k = \sum_{0<\sigma_l<1} \kappa_{l,k}^{\phi} s_l + v_k$, then we have the following estimates in the H^1 norm:

$$\|\phi - \phi_k\|_{H^1(\Omega)} \leq C h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1. \tag{5.5.71}$$

Proof. Let $\hat{\phi} \in H^1(\Omega)$ be the exact solution of the following problem:

$$(\epsilon^{-1} \boldsymbol{\nabla} \times \hat{\phi}, \boldsymbol{\nabla} \times v) = (\mu \xi_N, v) \quad \forall v \in H^1(\Omega), \tag{5.5.72a}$$

$$(\hat{\phi}, 1) = 0. \tag{5.5.72b}$$

Then $\hat{\phi}$ is an approximation of ϕ, which is the exact solution of the problem:

$$(\epsilon^{-1} \boldsymbol{\nabla} \times \phi, \boldsymbol{\nabla} \times v) = (\mu \xi, v) \quad \forall v \in H^1(\Omega), \tag{5.5.73a}$$

$$(\phi, 1) = 0. \tag{5.5.73b}$$

We denote the singular representation for $\hat{\phi}$ by

$$\hat{\phi} = \sum_{0<\sigma_l<1} \kappa_l^{\hat{\phi}} s_l + w_{\hat{\phi}}.$$

Because of the estimate (5.5.68) the error between ϕ and $\hat{\phi}$ can be estimated by

$$\|\phi - \hat{\phi}\|_{H^1(\Omega)} \leq C \|\xi - \xi_N\|_{L_2(\Omega)} \lesssim h_N^{1+\sigma_1-1} \|\boldsymbol{\nabla} \times \boldsymbol{f}\|_{L_2(\Omega)}. \tag{5.5.74}$$

Moreover, from (5.5.74) and the extraction formulas for κ_l^{ϕ} and $\kappa_l^{\hat{\phi}}$, we have

$$
\begin{aligned}
|\kappa_l^{\phi} - \kappa_l^{\hat{\phi}}| &\leq \frac{1}{2\sigma_1} \Big| \int_\Omega \mu(\xi - \xi_N)s_l^* + (\phi - \hat{\phi}_N)\Delta s_l^* dx \Big| \\
&\lesssim h_N^{1+\sigma_1-\epsilon} \|\boldsymbol{\nabla} \times \boldsymbol{f}\|_{L_2(\Omega)} \qquad (5.5.75)
\end{aligned}
$$

and

$$
\begin{aligned}
\|w_\phi - w_{\hat{\phi}}\|_{H^1(\Omega)} &\lesssim \|\xi - \xi_N\|_{L_2(\Omega)} + \sum_{0<\sigma_l<1} |\kappa_l^{\phi} - \kappa_l^{\hat{\phi}}| \\
&\lesssim h_N^{1+\sigma_1-1} \|\boldsymbol{\nabla} \times \boldsymbol{f}\|_{L_2(\Omega)}. \qquad (5.5.76)
\end{aligned}
$$

We know that ϕ_k is an approximation of $\hat{\phi}$. A similar argument to the proof of Theorem 5.5.3 gives the following estimates:

$$
\sum_{0<\sigma_l<1} |\kappa_l^{\hat{\phi}} - \kappa_{l,k}^{\phi}| \leq C h_k^{1+\sigma_1-\epsilon} \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1, \qquad (5.5.77)
$$

$$
\|w_{\hat{\phi}} - v_k\|_{H^1(\Omega)} \leq C h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1. \qquad (5.5.78)
$$

Let $\phi_k = \sum_{0<\sigma_l<1} \kappa_{l,k}^{\phi} s_l + v_k$, then we have the following estimates in the H^1 norm:

$$
\|\hat{\phi} - \phi_k\|_{H^1(\Omega)} \leq C h_k \|\nabla \times \boldsymbol{f}\|_{L_2(\Omega)} \quad \forall k \geq 1. \qquad (5.5.79)
$$

Now (5.5.74)-(5.5.79) imply (5.5.69)-(5.5.71). □

5.6 Numerical Results

In this section we present the results of several numerical tests that illustrate the performance of our algorithm. The first two numerical examples are performed on the L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1]^2$, where the subdomains are $\Omega_1 = (-1,0) \times (0,1)$, $\Omega_2 = (-1,0)^2$, and $\Omega_3 = (0,1) \times (-1,0)$ (See Figure 5.1). We use the P_1 finite element in the experiments. The mesh size h_k for the k^{th} level grid is taken to be $1/(3 \cdot 2^k)$. All the computations are done using a W-cycle k^{th} level iteration with 50 smoothing steps, and the number of nested iterations in both full

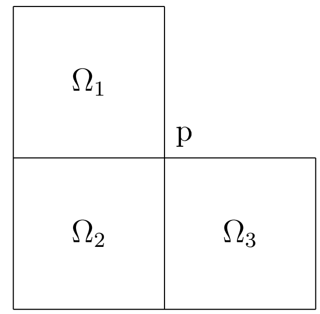

FIGURE 5.1. The domain Ω and its subdomains.

m	$k=4$	$k=5$	$k=6$	$k=7$
11	0.4662	0.6128	0.8082	0.9803
12	0.4461	0.5874	0.7528	0.9545
13	0.4286	0.5645	0.7028	0.9236
14	0.4131	0.5437	0.5824	0.8903
15	0.3993	0.5249	0.6416	0.8560
16	0.3870	0.5079	0.6218	0.8217

TABLE 5.1. Contraction numbers for the symmetric W-cycle algorithm on the L-shaped domain with m smoothing steps for the case $\epsilon = [1/350, 1, 1/350]$

multigrid algorithms (See Algorithm 5.4.1 and Algorithm 5.4.2 in Subsection 5.4.2) is also 50.

We will consider the case where $\epsilon = [1/350, 1, 1/350]$. In this situation, the contraction numbers for the symmetric W-cycle algorithm are given in Table 5.1.

Example 5.6.1. We solve equations (5.1.5), (5.1.3) on the domain Ω where $\alpha = 0$, $\epsilon = [1/350, 1, 1/350]$ and $\mu = [350, 1, 350]$. The vector function $\boldsymbol{f}$ is given by

$$\boldsymbol{f}(x) = \begin{bmatrix} r^{\sigma_1}\varrho'_{cut}(A_j \sin((\sigma_1+1)\theta) - B_j \cos((\sigma_1+1)\theta)) \\ r^{\sigma_1}\varrho'_{cut}(-A_j \cos((\sigma_1+1)\theta) - B_j \sin((\sigma_1+1)\theta)) \end{bmatrix}, \tag{5.6.1}$$

for x in the subdomain $\Omega_j, 1 \le j \le J$, where (r, θ) are the polar coordinates of the point $(0,0)$, $\lambda_1 = \sigma_1^2$ is the first eigenvalue of the related Sturm-Liouville problem (cf. Subsection 2.2.2), $\sigma_1 = 0.048066746316346...$, A_j, B_j are coefficients appearing in the eigenfunction Θ, and ϱ_{cut} is the cut-off function defined by

$$\varrho_{cut}(r) = \begin{cases} 1, & 0 \le r \le 1/4 \\ -192r^5 + 480r^4 - 440r^3 + 180r^2 - \frac{135}{4}r + \frac{27}{8}, & 1/4 < r < 3/4 \\ 0, & r \ge 3/4. \end{cases} \tag{5.6.2}$$

In this case the exact solution $\xi = s = r^{\sigma_1}\Theta(\theta)\varrho_{cut}$. The numerical results are tabulated in Table 5.2. For comparison, we solved the same problem by full multigrid without using the extraction formula. The numerical results are tabulated in Table 5.3.

From Table 5.2, we see that the approximate stress intensity factor κ_k^ξ is very accurate when using the new full multigrid with the extraction formula . Actually, the relative error between κ_8^ξ and the exact one $\kappa^\xi = 1$ is less than 0.1%. However, when using the full multigrid without the extraction formula, we see from Table 5.3 that the relative error between κ_8^ξ and the exact one $\kappa^\xi = 1$ is larger than 30%. Because the equation (5.1.3) has ξ as the right-hand side input function, we need a good approximation to ξ to obtain a good approximation to ϕ. Therefore, the numerical results tell us that the standard full multigrid method can not obtain a reliable approximation to ϕ, but the new full multigrid method with extraction formula has a much better performance than the standard and it can give us a reliable approximation to ϕ.

Furthermore we consider the error between the values of the exact solution and the numerical solution at a particular nodal point. Also, we consider the L_∞ error between the values of the exact solution and the numerical solution. Those errors provide another way to check the accuracy of our algorithm.

We choose two particular nodal points $(1/3, -1/3)$ and $(2/3, -2/3)$ and use the following notations to denote the errors between the values of the exact solution and the numerical solutions $\xi_k = w_k + \kappa_k^\xi s$ (obtained by full multigrid using the extraction formula) or $\xi_k = w_k$ (obtained by full multigrid without using the extraction formula) at the two points:

$$e_k^1 = |\xi_k(1/3, -1/3) - \xi(1/3, -1/3)| \tag{5.6.3}$$

and

$$e_k^2 = |\xi_k(2/3, -2/3) - \xi(2/3, -2/3)|. \tag{5.6.4}$$

The k^{th} level convergence rates are computed and denoted by:

$$\eta_k^1 = \log_2(\frac{e_k^1}{e_{k+1}^1}) \tag{5.6.5}$$

and

$$\eta_k^2 = \log_2(\frac{e_k^2}{e_{k+1}^2}). \tag{5.6.6}$$

We also denote the errors in L_∞ norm and their convergence rates by:

$$e_k^\infty = \|\xi_k - \Pi_h \xi\|_\infty = \max_{p \text{ is a nodal point}} |\xi_k(p) - \xi(p)| \tag{5.6.7}$$

and

$$\eta_k^\infty = \log_2(\frac{e_k^\infty}{e_{k+1}^\infty}), \tag{5.6.8}$$

where Π_h is the nodal interpolant (cf. Subsection 2.4). The numerical results are tabulated in Table 5.4 and Table 5.5. From Table 5.4 we clearly see that the numerical solutions obtained by the full multigrid methods using the extraction formulas converge quickly. Meanwhile, from the results in Table 5.5 we see that the numerical solutions obtained by the full multigrid methods without using the extraction formulas converge very slowly.

Example 5.6.2. We solve equations (5.1.5) and (5.1.3) on the domain Ω where $\alpha = 1$, $\epsilon = [1/350, 1, 1/350]$, $\mu = [1, 1, 1]$, and the right-hand side vector function

k	$\frac{\|w_{k+1}-w_k\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order	$\frac{\|w_{k+1}-w_k\|_{H^1}}{\|\boldsymbol{f}\|_{L_2}}$	Order	κ_k^{ξ}	Order
k= 3	4.13089E-002	2.32	9.54251E-002	2.27	1.33663	2.40
k= 4	8.25511E-003	2.70	1.97396E-002	2.65	1.06363	1.65
k= 5	1.26805E-003	1.75	3.13929E-003	1.69	1.02034	1.55
k= 6	3.76819E-004	1.60	9.71532E-004	1.53	1.00695	1.58
k= 7	1.24339E-004	-	3.35502E-004	-	1.00233	1.63
k= 8	-	-	-	-	1.00075	-
k	$\frac{\|v_{k+1}-v_k\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order	$\frac{\|v_{k+1}-v_k\|_{H^1}}{\|\boldsymbol{f}\|_{L_2}}$	Order	κ_k^{ϕ}	Order
k= 3	4.97390E-002	2.24	1.12834E-001	2.08	1.55132	0.54
k= 4	1.05195E-002	0.04	2.65966E-002	0.04	1.87351	0.96
k= 5	1.02509E-002	0.88	2.59071E-002	0.81	2.21573	1.13
k= 6	5.58506E-003	1.17	1.48107E-002	1.08	2.41218	1.18
k= 7	2.48641E-003	-	6.99230E-003	-	2.50392	1.05
k= 8	-	-	-	-	2.54140	-

TABLE 5.2. Results of the full multigrid method with exaction formulas for Example 5.6.1

k	$\frac{\|w_{k+1}-w_k\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order	$\frac{\|w_{k+1}-w_k\|_{H^1}}{\|\boldsymbol{f}\|_{L_2}}$	Order	κ_k^{ξ}	Order
k= 3	1.60827E-003	0.15	1.36642E-002	0.79	1.33663	0.83
k= 4	1.45351E-003	0.05	7.89844E-003	0.54	0.81059	-0.76
k= 5	1.40607E-003	0.03	5.48214E-003	0.22	0.68034	-0.10
k= 6	1.37394E-003	0.03	4.69297E-003	0.04	0.65850	0.02
k= 7	1.34356E-003	-	4.54985E-003	-	0.66423	0.05
k= 8	-	-	-	-	0.67665	-
h	$\frac{\|v_{k+1}-v_k\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order	$\frac{\|v_{k+1}-v_k\|_{H^1}}{\|\boldsymbol{f}\|_{L_2}}$	Order	κ_k^{ϕ}	Order
k= 3	8.76793E-003	0.02	2.39732E-002	-0.06	-3.6208	-
k= 4	8.61832E-003	0.03	2.49235E-002	-0.05	-3.7190	-
k= 5	8.46529E-003	0.03	2.57451E-002	-0.04	-3.8143	-
k= 6	8.30104E-003	0.03	2.64454E-002	-0.03	-3.9076	-
k= 7	8.12447E-003	-	2.70319E-002	-	-3.9989	-
k= 8	-	-	-	-	-4.0882	-

TABLE 5.3. Results of the full multigrid method without extraction formula for Example 5.6.1

k	e_k^1	η_k^1	e_k^2	η_k^2	e_k^∞	η_k^∞
k= 3	2.44230E-002	1.64	2.44371E-002	1.64	2.49002E-002	1.64
k= 4	7.87278E-003	2.46	7.85942E-002	2.46	7.96934E-003	1.45
k= 5	1.42442E-003	1.71	1.42349E-003	1.71	1.45365E-003	1.70
k= 6	4.36182E-004	1.61	4.35866E-004	1.61	4.48301E-004	1.60
k= 7	1.42605E-004	1.64	1.42499E-004	1.64	1.47737E-004	1.63
k= 8	4.58088E-005	-	4.57748E-005	-	4.78813E-005	-

TABLE 5.4. Pointwise errors for Example 5.6.1 using the full multigrid algorithms with extraction formulas, where e_k^1, η_k^1, e_k^2, η_k^2, e_k^∞ and η_k^∞ are defined by (5.6.3)–(5.6.8)

k	e_k^1	η_k^1	e_k^2	η_k^2	e_k^∞	η_k^∞
k= 3	2.44230E-002	-	2.44371E-002	-	2.49002E-002	-
k= 4	2.33570E-002	-	2.33482E-002	-	2.36740E-002	-
k= 5	2.23848E-002	-	2.23700E-002	-	2.28435E-002	-
k= 6	2.14447E-002	-	2.14313E-002	-	2.20428E-002	-
k= 7	2.05315E-002	-	2.05161E-002	-	2.12710E-002	-
k= 8	1.96366E-002	-	1.96218E-002	-	2.05272E-002	-

TABLE 5.5. Pointwise errors for Example 5.6.1 using the full multigrid algorithms without extraction formulas, where e_k^1, η_k^1, e_k^2, η_k^2, e_k^∞ and η_k^∞ are defined by (5.6.3)–(5.6.8)

is given by

$$\boldsymbol{f}(x) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if } x \in \Omega_1 \text{ or } \Omega_2 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{otherwise.} \end{cases} \tag{5.6.9}$$

The numerical results are tabulated in Table 5.6. For comparison we solved the same problem by full multigrid without using the extraction formula. The numerical results are tabulated in Table 5.7.

Comparing Table 5.6 and Table 5.7, we can clearly see the improvement of the order of convergence for κ_k^ξ and w_k while using the algorithm with extraction formulas. The order of convergence for κ_k^ξ and w_k matches the estimates (5.5.32) and (5.5.33).

k	$\frac{\|w_{k+1}-w_k\|_{L^2}}{\|\boldsymbol{f}\|_{L^2}}$	Order	$\frac{\|w_{k+1}-w_k\|_{H^1}}{\|\boldsymbol{f}\|_{L^2}}$	Order	κ_k^{ξ}	Order
k= 3	1.30236E-002	2.32	8.84351E-002	0.96	0.40726	0.54
k= 4	2.60059E-003	0.01	4.54224E-002	0.87	0.48641	0.99
k= 5	2.58557E-003	0.91	2.48881E-002	0.91	0.57346	1.17
k= 6	1.37815E-003	1.20	1.32454E-002	0.94	0.62234	1.20
k= 7	5.99051E-004	-	6.91438E-003	-	0.64457	1.05
k= 8	-	-	-	-	0.65340	-
h	$\frac{\|v_{k+1}-v_k\|_{L^2}}{\|\boldsymbol{f}\|_{L^2}}$	Order	$\frac{\|v_{k+1}-v_k\|_{H^1}}{\|\boldsymbol{f}\|_{L^2}}$	Order	κ_k^{ϕ}	Order
k= 3	1.26544E-002	2.20	2.91030E-002	1.97	-0.39397	0.55
k= 4	2.75549E-003	0.06	7.34879E-003	0.11	-0.47836	0.96
k= 5	2.64996E-003	0.88	6.80194E-003	0.81	-0.56680	1.13
k= 6	1.44195E-003	1.17	3.86959E-003	1.08	-0.61752	1.18
k= 7	6.41671E-003	-	1.82865E-003	-	-0.64119	1.05
k= 8	-	-	-	-	-0.65086	-

TABLE 5.6. Results of the full multigrid method with extraction formulas for Example 5.6.2

k	$\frac{\|w_{k+1}-w_k\|_{L^2}}{\|\boldsymbol{f}\|_{L^2}}$	Order	$\frac{\|w_{k+1}-w_k\|_{H^1}}{\|\boldsymbol{f}\|_{L^2}}$	Order	κ_k^{ξ}	Order
k= 3	1.65040E-003	0.69	8.33942E-002	0.89	0.40726	-0.05
k= 4	1.02427E-003	0.12	4.50228E-002	0.90	0.41681	0.01
k= 5	9.40852E-004	0.04	2.41734E-002	0.89	0.42670	0.03
k= 6	9.14799E-004	0.04	1.30572E-002	0.83	0.43647	0.03
k= 7	8.92539E-004	-	7.34235E-003	-	0.44611	-
k= 8	-	-	-	-	0.45551	-
h	$\frac{\|v_{k+1}-v_k\|_{L^2}}{\|\boldsymbol{f}\|_{L^2}}$	Order	$\frac{\|v_{k+1}-v_k\|_{H^1}}{\|\boldsymbol{f}\|_{L^2}}$	Order	κ_k^{ϕ}	Order
k= 3	1.00389E-003	0.06	5.63034E-002	0.59	-0.40462	-0.05
k= 4	9.65823E-004	0.03	3.73602E-003	0.26	-0.41468	0.01
k= 5	9.44524E-004	0.03	3.13045E-003	0.06	-0.42507	0.02
k= 6	9.25241E-004	0.03	3.01696E-003	-0.01	-0.43541	0.03
k= 7	9.05325E-004	-	3.02828E-003	-	-0.44557	-
k= 8	-	-	-	-	-0.45551	-

TABLE 5.7. Results of the full multigrid method without extraction formulas for Example 5.6.2

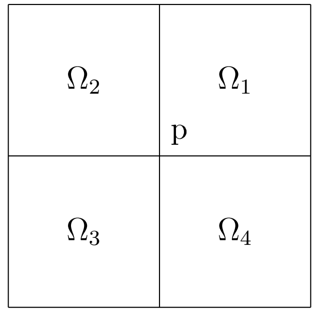

FIGURE 5.2. The domain Ω and its subdomains.

The last numerical example is performed on the domain $\Omega = (-1,1)^2$, where the subdomains are $\Omega_1 = (0,1) \times (0,1)$, $\Omega_2 = (-1,0) \times (0,1)$, $\Omega_3 = (-1,0)^2$, and $\Omega_4 = (0,1) \times (-1,0)$ (See Figure 5.2). We use the P_1 finite element in the experiments. The mesh size h_k for the k^{th} level grid is taken to be $1/2^{k-1}$. All the computations are done using a W-cycle k^{th} level iteration with 50 smoothing steps, and the number of nested iterations in both full multigrid algorithms (See Algorithm 5.4.1 and Algorithm 5.4.2 in Subsection 5.4.2) is also 50.

Example 5.6.3. We solve equations (5.1.5) and (5.1.3) on the domain Ω where $\alpha = 1$, $\epsilon = [1/10, 1/10^3, 1, 1/10^4]$, $\mu = [1,1,1,1]$, and the right-hand side vector function is given by

$$\boldsymbol{f}(x) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if } x \in \Omega_1 \text{ or } \Omega_2 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{otherwise.} \end{cases} \tag{5.6.10}$$

In this case, the square root of the first eigenvalue of the related Sturm-Liouville problem (cf. Subsection 2.2.2) is $\sigma_1 = 0.069817020390924...$ The numerical results are tabulated in Table 5.8. For comparison we solved the same problem by full multigrid without using the extraction formula. The numerical results are tabulated in Table 5.9.

k	$\frac{\|w_{k+1}-w_k\|_{L^2}}{\|\boldsymbol{f}\|_{L^2}}$	Order	$\frac{\lvert w_{k+1}-w_k\rvert_{H^1}}{\|\boldsymbol{f}\|_{L^2}}$	Order	κ_k^{ξ}	Order
k=4	2.50818e-002	1.93	1.04075e-001	0.92	1.57342	1.92
k=5	6.59066e-003	0.66	5.48533e-002	0.53	1.99063	0.55
k=6	4.16937e-003	1.07	3.79186e-002	0.69	2.27528	1.01
k=7	1.98659e-003	1.32	2.35356e-002	0.77	2.41677	1.29
k=8	7.97897e-004	-	1.37912e-002	-	2.47466	1.49
k=9	-	-	-	-	2.49524	-
h	$\frac{\|v_{k+1}-v_k\|_{L^2}}{\|\boldsymbol{f}\|_{L^2}}$	Order	$\frac{\lvert v_{k+1}-v_k\rvert_{H^1}}{\|\boldsymbol{f}\|_{L^2}}$	Order	κ_k^{ϕ}	Order
k=4	2.63827e-003	1.83	6.47826e-003	1.02	-0.13092	1.89
k=5	7.39617e-004	0.65	3.19478e-003	0.33	-0.16625	0.56
k=6	4.71923e-004	1.11	2.54380e-003	0.79	-0.19026	1.01
k=7	2.18638e-004	1.42	1.47385e-003	1.01	-0.20221	1.29
k=8	8.15293e-005	-	7.32761e-004	-	-0.20710	1.49
k=9	-	-	-	-	-0.20885	-

TABLE 5.8. Results of the full multigrid method with extraction formulas for Example 5.6.3

Comparing Table 5.8 and Table 5.9, we can clearly see the improvement of the order of convergence for κ_k^{ξ} and w_k while using the algorithm with extraction formulas. The order of convergence for κ_k^{ξ} and w_k matches the estimates (5.5.32) and (5.5.33).

k	$\frac{\|w_{k+1}-w_k\|_{L^2}}{\|\boldsymbol{f}\|_{L^2}}$	Order	$\frac{\lvert w_{k+1}-w_k\rvert_{H^1}}{\|\boldsymbol{f}\|_{L^2}}$	Order	κ_k^{ξ}	Order
k=4	3.43595e-003	0.40	7.52331e-002	0.55	1.57342	4.67
k=5	2.60679e-003	0.17	5.12986e-002	0.51	1.63532	0.16
k=6	2.32044e-003	0.11	3.60078e-002	0.59	1.69081	0.07
k=7	2.15047e-003	0.08	2.39613e-002	0.56	1.74351	0.06
k=8	2.04033e-003	-	1.62005e-002	-	1.79395	0.06
k=9	-	-	-	-	1.84220	-
h	$\frac{\|v_{k+1}-v_k\|_{L^2}}{\|\boldsymbol{f}\|_{L^2}}$	Order	$\frac{\lvert v_{k+1}-v_k\rvert_{H^1}}{\|\boldsymbol{f}\|_{L^2}}$	Order	κ_k^{ϕ}	Order
k=4	3.35034e-004	-0.03	2.35234e-003	-0.27	-0.13441	4.59
k=5	3.41577e-004	0.09	2.82980e-003	0.22	-0.13997	0.21
k=6	3.20896e-004	0.18	2.42614e-003	0.62	-0.14479	0.09
k=7	2.82526e-004	0.13	1.57991e-003	0.58	-0.14934	0.06
k=8	2.57405e-004	-	1.05698e-003	-	-0.15369	0.06
k=9	-	-	0.00000e+000	-	-0.15787	-

TABLE 5.9. Results of the full multigrid method without extraction formulas for Example 5.6.3

Appendix A: Proof for Theorem 2.4.7

Let C denote a generic constant independent of the mesh size h.

Let $s = r^\sigma \sin(\sigma\theta)$, where (r,θ) are the polar coordinates with respect to a corner c and σ is a number between 0 and 1.

Around the corner c, we have

$$\begin{aligned}\sum_{T\in\mathscr{T}_h, c\in\bar{T}} \|s - \Pi_h s\|^2_{L_2(T)} &\leq \sum_{T\in\mathscr{T}_h, c\in\bar{T}} \int_T (2s^2 + 2|\Pi_h s|^2)dx \\ &= 2\sum_{T\in\mathscr{T}_h, c\in\bar{T}} \int_T s^2 dx + 2\sum_{T\in\mathscr{T}_h, c\in\bar{T}} \int_T |\Pi_h s|^2 dx \qquad (5.6.11)\end{aligned}$$

and

$$\begin{aligned}\sum_{T\in\mathscr{T}_h, c\in\bar{T}} \|\boldsymbol{\nabla} s - \boldsymbol{\nabla}\Pi_h s\|^2_{L_2(T)} &\leq \sum_{T\in\mathscr{T}_h, c\in\bar{T}} \int_T (2|\boldsymbol{\nabla} s|^2 + 2|\boldsymbol{\nabla}\Pi_h s|^2)dx \\ &= 2\sum_{T\in\mathscr{T}_h, c\in\bar{T}} \int_T |\boldsymbol{\nabla} s|^2 dx + 2\sum_{T\in\mathscr{T}_h, c\in\bar{T}} \int_T |\boldsymbol{\nabla}\Pi_h s|^2 dx. \qquad (5.6.12)\end{aligned}$$

For a triangle T around the corner c, i.e. $c\in\bar{T}$, let $x_j, j = 1,2,3$ be the coordinates of the vertices of the triangle T, and $e_j, j = 1,2,3$ be the local basis functions. Then we have

$$\int_T s^2 dx \leq \int_T r^{2\sigma} dx \leq C\int_0^h r^{2\sigma+1} dr \leq Ch^{2\sigma+2}, \qquad (5.6.13)$$

$$\begin{aligned}\int_T |\Pi_h s|^2 dx &= \int_T (\sum_{j=1}^3 s(x_j)e_j)^2 dx \leq \int_T (\sum_{j=1}^3 s(x_j)^2)(\sum_{j=1}^3 e_j^2) dx \\ &\leq Ch^{2\sigma}\int_T (\sum_{j=1}^3 e_j^2) dx \leq Ch^{2\sigma+2}, \qquad (5.6.14)\end{aligned}$$

$$\int_T |\boldsymbol{\nabla} s|^2 dx \leq C\int_T r^{2\sigma-2} dx \leq C\int_0^h r^{2\sigma-1} dr \leq Ch^{2\sigma}, \qquad (5.6.15)$$

and

$$\int_T |\boldsymbol{\nabla}\Pi_h s|^2 dx = \int_T |\sum_{j=1}^3 s(x_j)\boldsymbol{\nabla} e_j|^2 dx \leq \int_T (\sum_{j=1}^3 s(x_j)^2)(\sum_{j=1}^3 |\boldsymbol{\nabla} e_j|^2) dx$$

$$\leq Ch^{2\sigma} \int_T (\sum_{j=1}^{3} |\boldsymbol{\nabla} e_j|^2) dx \leq Ch^{2\sigma}. \tag{5.6.16}$$

By (5.6.11), (5.6.13) and (5.6.14), we have

$$\sum_{T \in \mathscr{T}_h, c \in \bar{T}} \|s - \Pi_h s\|_{L_2(T)}^2 \leq Ch^{2\sigma+2}. \tag{5.6.17}$$

By (5.6.12), (5.6.15) and (5.6.16), we have

$$\sum_{T \in \mathscr{T}_h, c \in \bar{T}} \|\boldsymbol{\nabla} s - \boldsymbol{\nabla} \Pi_h s\|_{L_2(T)}^2 \leq Ch^{2\sigma}. \tag{5.6.18}$$

Let $\Omega_h = \{x \in \Omega : |x - c| > \delta h\}$ for some δ between 0 and 1 so that

$$T \subset \Omega_h \text{ if } T \in \mathscr{T}_h, c \notin \bar{T}.$$

Therefore,

$$\sum_{T \in \mathscr{T}_h, c \notin \bar{T}} \|s - \Pi_h s\|_{L_2(T)}^2 \leq \|s - \Pi_h s\|_{L_2(\Omega_h)}^2, \tag{5.6.19}$$

and

$$\sum_{T \in \mathscr{T}_h, c \notin \bar{T}} \|s - \Pi_h s\|_{H^1(T)}^2 \leq \|s - \Pi_h s\|_{H^1(\Omega_h)}^2. \tag{5.6.20}$$

Since $s \in H^2(\Omega_h)$, by [20, Theorem (4.4.4)], we have

$$\|s - \Pi_h s\|_{L_2(\Omega_h)} + h|s - \Pi_h s|_{H^1(\Omega_h)} \leq Ch^2 |s|_{H^2(\Omega_h)}. \tag{5.6.21}$$

Note that

$$|s|_{H^2(\Omega_h)}^2 \leq C \int_{\Omega_h} r^{2\sigma-4} dx \leq C \int_h^R r^{2\sigma-3} dr \leq Ch^{2\sigma-2}, \tag{5.6.22}$$

where R is the diameter of the domain Ω.

Therefore (5.6.21) and (5.6.22) imply

$$\|s - \Pi_h s\|_{L_2(\Omega_h)} + h|s - \Pi_h s|_{H^1(\Omega_h)} \leq Ch^{\sigma+1}. \tag{5.6.23}$$

By (5.6.19), (5.6.20), and (5.6.23) we have

$$\sum_{T\in\mathscr{T}_h, c\notin\bar{T}} \|s-\Pi_h s\|^2_{L_2(T)} \le Ch^{2\sigma+2}, \tag{5.6.24}$$

and

$$\sum_{T\in\mathscr{T}_h, c\notin\bar{T}} \|s-\Pi_h s\|^2_{H^1(T)} \le Ch^{2\sigma}. \tag{5.6.25}$$

Now by (5.6.17), (5.6.18), (5.6.24), and (5.6.25), we have

$$\|s-\Pi_h s\|_{L_2(\Omega)} + h|s-\Pi_h s|_{H^1(\Omega)} \le Ch^{\sigma+1}. \tag{5.6.26}$$

If $\beta = 1$, then $u \in H^2(\Omega)$. So (2.4.8) is true by [20, Theorem (4.4.4)].

If $\beta < 1$, then, from Section 2.2.1, we know that

$$u = u_S + u_R, \tag{5.6.27}$$

where $u_R \in H^2(\Omega)$ and $u_S = \sum_{\omega_l>\pi} \kappa_l s_l$. The regular part $u_R \in H^2(\Omega)$ and hence

$$\|u_R-\Pi_h u_R\|_{L_2(\Omega)} + h|u_R-\Pi_h u_R|_{H^1(\Omega)} \le Ch^2\|u_R\|_{H^2(\Omega)}, \tag{5.6.28}$$

by [20, Theorem (4.4.4)].

For $s_i = r^{\pi/\omega_i}\sin((\pi/\omega_i)\theta)\varrho_{cut}$ with $\omega_i > \pi$, we have

$$s_i := r^{\pi/\omega_i}\sin(\pi/\omega_i)\theta)(\varrho_{cut}-1) + r^{\pi/\omega_i}\sin(\pi/\omega_i)\theta).$$

Let $s_{i,1} = r^{\pi/\omega_i}\sin(\pi/\omega_i)\theta)(\varrho_{cut}-1)$ and $s_{i,2} = r^{\pi/\omega_i}\sin((\pi/\omega_i)\theta)$. Then $s_{i,1} \in H^2(\Omega)$, so

$$\|s_{i,1}-\Pi_h s_{i,1}\|_{L_2(\Omega)} + h|s_{i,1}-\Pi_h s_{i,1}|_{H^1(\Omega)} \le Ch^2\|s_{i,1}\|_{H^2(\Omega)} \le Ch^2. \tag{5.6.29}$$

By (5.6.26), we have

$$\|s_{i,2}-\Pi_h s_{i,2}\|_{L_2(\Omega)} + h|s_{i,2}-\Pi_h s_{i,2}|_{H^1(\Omega)} \le Ch^{\sigma+1}. \tag{5.6.30}$$

Now (5.6.27), (5.6.28), (5.6.29), and (5.6.30) together imply (2.4.9).

Appendix B: Proof for (2.2.17)

We will follow the methodology of Example 1 in [54, Section 5] and consider a general Sturm-Liouville problem:

$$\Theta''(\theta) + \sigma^2\Theta(\theta) = 0 \quad \text{for} \quad \theta_{i-1} < \theta < \theta_i, \quad i = 1, ..., n, \tag{5.6.31a}$$

$$\Theta'(\theta_0) = \Theta'(\theta_n) = 0, \tag{5.6.31b}$$

$$\Theta(\theta_i-) = \Theta(\theta_i+) \quad \text{for} \quad i = 1, ..., n-1, \tag{5.6.31c}$$

$$\rho_{i+1}\Theta'(\theta_i+) = \rho_i\Theta'(\theta_i-) \quad \text{for} \quad i = 1, ..., n-1, \tag{5.6.31d}$$

where $0 \leq \theta_0 < \theta_1 < ... < \theta_n \leq 2\pi$, $\theta_n - \theta_0 \neq 2\pi$, and $\rho_i > 0 \quad \text{for } i = 1, 2, ..., n$.

Our goal is to find the eigenvalues $\lambda = \sigma^2$ of (5.6.31).

The solutions of (5.6.31a) have the general form

$$\Theta(\theta) = A_i \cos(\sigma\theta) + B_i \sin(\sigma\theta) \quad \text{for } \theta_{i-1} \leq \theta \leq \theta_i, i = 1, ..., n.$$

Substituting the general solution of Θ into the boundary condition (5.6.31b) and the interface conditions (5.6.31c)-(5.6.31d), we obtain a linear system about the variables A_i and B_i. Denote the determinant of the coefficient matrix of this linear system by $D_n^N(\sigma)$. We also denote, by the symbol $D_n^M(\sigma)$, the determinant of the coefficient matrix of the linear system obtained from the Sturm-Liouville problem with mixed boundary condition on the external boundary, i.e., we replace the Neumann boundary condition $\Theta'(\theta_n) = 0$ in (5.6.31b) by the Dirichlet boundary condition $\Theta(\theta_n) = 0$. When $n = 1$, the determinants $D_1^N(\sigma)$ and $D_1^M(\sigma)$ are given by

$$D_1^N = \begin{vmatrix} -\sin(\sigma\theta_0) & \cos(\sigma\theta_0) \\ -\sin(\sigma\theta_1) & \cos(\sigma\theta_1) \end{vmatrix} \tag{5.6.32}$$

and

$$D_1^M = \begin{vmatrix} -\sin(\sigma\theta_0) & \cos(\sigma\theta_0) \\ \cos(\sigma\theta_1) & \sin(\sigma\theta_1) \end{vmatrix}. \tag{5.6.33}$$

From (5.6.32) and (5.6.33), a simple computation implies

$$D_1^N(\sigma) = \sin(\sigma\omega_1) \tag{5.6.34a}$$

and

$$D_1^M(\sigma) = -\cos(\sigma\omega_1), \tag{5.6.34b}$$

where $\omega_1 = \theta_1 - \theta_0$.

When $n = 2$, the determinants $D_2^N(\sigma)$ and $D_2^M(\sigma)$ are given by

$$D_2^N = \begin{vmatrix} -\sin(\sigma\theta_0) & \cos(\sigma\theta_0) & 0 & 0 \\ \cos(\sigma\theta_1) & \sin(\sigma\theta_1) & -\cos(\sigma\theta_1) & -\sin(\sigma\theta_1) \\ -\rho_1\sin(\sigma\theta_1) & \rho_1\cos(\sigma\theta_1) & \rho_2\sin(\sigma\theta_1) & -\rho_2\cos(\sigma\theta_1) \\ 0 & 0 & -\sin(\sigma\theta_2) & \cos(\sigma\theta_2) \end{vmatrix} \tag{5.6.35}$$

and

$$D_2^M = \begin{vmatrix} -\sin(\sigma\theta_0) & \cos(\sigma\theta_0) & 0 & 0 \\ \cos(\sigma\theta_1) & \sin(\sigma\theta_1) & -\cos(\sigma\theta_1) & -\sin(\sigma\theta_1) \\ -\rho_1\sin(\sigma\theta_1) & \rho_1\cos(\sigma\theta_1) & \rho_2\sin(\sigma\theta_1) & -\rho_2\cos(\sigma\theta_1) \\ 0 & 0 & \cos(\sigma\theta_2) & \sin(\sigma\theta_2) \end{vmatrix}. \tag{5.6.36}$$

Using expansion by minors with respect to the last row of $D_2^N(\sigma)$, $D_2^N(\sigma)$ can be computed by

$$D_2^N(\sigma) = \cos(\sigma\theta_2) * N_{44} + \sin(\sigma\theta_2) * N_{43}, \tag{5.6.37}$$

where

$$N_{44} = \begin{vmatrix} -\sin(\sigma\theta_0) & \cos(\sigma\theta_0) & 0 \\ \cos(\sigma\theta_1) & \sin(\sigma\theta_1) & -\cos(\sigma\theta_1) \\ -\rho_1\sin(\sigma\theta_1) & \rho_1\cos(\sigma\theta_1) & \rho_2\sin(\sigma\theta_1) \end{vmatrix} \tag{5.6.38}$$

and

$$N_{43} = \begin{vmatrix} -\sin(\sigma\theta_0) & \cos(\sigma\theta_0) & 0 \\ \cos(\sigma\theta_1) & \sin(\sigma\theta_1) & -\sin(\sigma\theta_1) \\ -\rho_1\sin(\sigma\theta_1) & \rho_1\cos(\sigma\theta_1) & -\rho_2\cos(\sigma\theta_1) \end{vmatrix}. \tag{5.6.39}$$

Using expansion by minors with respect to the last column of N_{43} and the last column of N_{44}, we obtain

$$N_{44} = \rho_2\sin(\sigma\theta_1)D_1^M(\sigma) + \rho_1\cos(\sigma\theta_1)D_1^N(\sigma) \tag{5.6.40}$$

and

$$N_{43} = -\rho_2\cos(\sigma\theta_1)D_1^M(\sigma) + \rho_1\sin(\sigma\theta_1)D_1^N(\sigma) \tag{5.6.41}$$

By (5.6.37), (5.6.41), (5.6.40) and trigonometry identity, we have

$$D_2^N(\sigma) = -\rho_2\sin(\sigma\omega_2)D_1^M(\sigma) + \rho_1\cos(\sigma\omega_2)D_1^N(\sigma), \tag{5.6.42}$$

where $\omega_2 = \theta_2 - \theta_1$. Similarly, we have

$$D_2^M(\sigma) = \rho_2\cos(\sigma\omega_2)D_1^M(\sigma) + \rho_1\sin(\sigma\omega_2)D_1^N(\sigma). \tag{5.6.43}$$

When $n \geq 3$, similar to the case when $n = 2$, we apply expansion by minors with respect to the last row of $D_n^N(\sigma)$ and the last row of $D_n^M(\sigma)$. We find that $D_n^N(\sigma)$ can be determined by the following recurrence formula:

$$D_n^N(\sigma) = -\rho_n\sin(\sigma\omega_n)D_{n-1}^M(\sigma) + \rho_{n-1}\cos(\sigma\omega_n)D_{n-1}^N(\sigma), \tag{5.6.44a}$$

$$D_n^M(\sigma) = \rho_n\cos(\sigma\omega_n)D_{n-1}^M(\sigma) + \rho_{n-1}\sin(\sigma\omega_n)D_{n-1}^N(\sigma). \tag{5.6.44b}$$

Here $\omega_n = \theta_n - \theta_{n-1}$. The square of the solutions of the equations $D_n^N(\sigma) = 0$ are the eigenvalues for the Sturm-Liouville problem (5.6.30).

For Example 2.2.4 in Subsection 2.2.2, we have $n = 3$, $\omega_1 = \omega_2 = \omega_3 = \pi/2$. So using the recurrence formulas (5.6.34) and (5.6.44), we have

$$\rho_2(\rho_1+\rho_2+\rho_3)\sin(\sigma\omega) - (\rho_2(\rho_1+\rho_2+\rho_3)+\rho_1\rho_3)\sin^3(\sigma\omega) = 0, \tag{5.6.45}$$

where $\omega = \pi/2$. For $0 < \sigma < 1$, we have that $0 < \sin(\sigma\omega) < 1$ and hence (5.6.45) implies that (2.2.17) holds.

References

[1] R. Adams and J. Fournier. *Sobolev Spaces (Second Edition).* Academic Press, 2003.

[2] I. Babuska and A. Miller. The post-processing approach in the finite element method - part 2: The calculation of stress intensity factors. *International Journal for Numerical Methods in Engineering*, 20:1111–1129, 1984.

[3] R.E. Bank. A comparison of two multilevel iterative methods for nonsymmetric and indefinite elliptic finite element equations. *SIAM Journal on Numerical Analysis*, 18:724–743, 1981.

[4] R.E. Bank and T.F. Dupont. An optimal order process for solving finite element equations. *Mathematics of Computation*, 36:35–51, 1981.

[5] H. Blum and M. Dobrowolski. On finite element methods for elliptic equations on domains with corners. *Computing*, 28:53–63, 1982.

[6] M. Blumenfeld. The regularity of interface-problems on corner-regions. *Lecture Notes in Mathematics*, 1121:38–54, 1985.

[7] A. Bossavit. *Computational Electromagnetism.* Academic Press, 1998.

[8] J.H. Bramble, Z. Leyk, and J. Pasciak. Iterative schemes for nonsymmetric and indefinite elliptic boundary value problems. *Mathematics of Computation*, 60:1–22, 1993.

[9] J.H. Bramble, J. Pasciak, and J. Xu. The analysis of multigrid algorithms for nonsymmetric and indefinite elliptic problems. *Mathematics of Computation*, 51:389–414, 1988.

[10] J.H. Bramble and J.E. Pasciak. New convergence estimates for multigrid algorithms. *Mathematics of Computation*, 49:311–329, 1987.

[11] J.H. Bramble and X. Zhang. *The analysis of multigrid methods, in: Ciarlet, P.G. and Lions, J.L. (Eds.), Handbook of Numerical Analysis VII.* Elsevier Science, 2000.

[12] S.C. Brenner. Overcoming corner singularities by multigrid methods. *SIAM Journal on Numerical Analysis*, 35:1883–1892, 1998.

[13] S.C Brenner, J. Cui, F. Li, and L.-Y. Sung. A nonconforming finite element method for a two-dimensional curl-curl and grad-div problem. *Numerische Mathematik*, 109:509–533, 2008.

[14] S.C. Brenner, J. Cui, Z. Nan, and L.-Y. Sung. Hodge decomposition for divergence-free vector fields and two-dimensional Maxwell's equations. *Mathematics of Computation*, 81:643–659, 2012.

[15] S.C Brenner, J. Cui, and L.-Y. Sung. An interior penalty method for a two dimensional curl-curl and grad-div problem. *Australian and New Zealand Industrial and Applied Mathematics Journal*, 50:C947–C975, 2009.

[16] S.C. Brenner, J. Gedicke, and L.-Y. Sung. An adaptive P_1 finite element method for two-dimensional Maxwell's equations. *Journal of Scientific Computing*, DOI:10.1007/s10915–012–9658–8, 2012.

[17] S.C. Brenner, F. Li, and L.-Y. Sung. A locally divergence-free nonconforming finite element method for the reduced time-harmonic Maxwell equations. *Mathematics of Computation*, 76:573–595, 2007.

[18] S.C. Brenner, F. Li, and L.-Y. Sung. A locally divergence-free interior penalty method for two-dimensional curl-curl problems. *SIAM Journal on Numerical Analysis*, 46:1190–1211, 2008.

[19] S.C. Brenner, F. Li, and L.-Y. Sung. A nonconforming penalty method for a two-dimensional curl-curl problem. *Mathematical Models and Methods in Applied Sciences*, 19, 2009.

[20] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, 2008.

[21] S.C. Brenner and L.-Y. Sung. Multigrid methods for the computation of singular solutions and stress intensity factors II: Crack singularities. *BIT*, 37:623–643, 1997.

[22] S.C. Brenner and L.-Y. Sung. Multigrid methods for the computation of singular solutions and stress intensity factors I: Corner singularities. *Mathematics of Computation*, 68:559–583, 1999.

[23] S.C. Brenner and L.-Y. Sung. Multigrid methods for the computation of singular solutions and stress intensity factors III: Interface singularities. *Computer Methods in Applied Mechanics and Engineering*, 192:4687–4702, 2003.

[24] W.L. Briggs, V.E. Henson, and S.F. McCormick. *A Multigrid Tutorial*. SIAM, 2000.

[25] P.G. Ciarlet. *The Finite Element Methods for Elliptic Problems*. North-Holland, 1978.

[26] P. Clément. Approximation by finite element functions using local regularization. *R.A.I.R.O.*, 2:77–84, 1975.

[27] M. Costabel and M. Dauge. Weighted regularization of Maxwell equations in polyhedral domains. *Numerische Mathematik*, 93:239–277, 2002.

[28] M. Costabel, M. Dauge, and S. Nicaise. Singularities of Maxwell interface problems. *Mathematical Modelling and Numerical Analysis*, 33:627–649, 1999.

[29] M. Dauge. *Elliptic Boundary Value Problems on Corner Domains*. Springer-Verlag, 1988.

[30] M. Dauge, M.-S. Lubuma, and S. Nicaise. Coefficients des singularitiés pour le probléme de Dirichlet sur un polygone. *Comptes Rendus de I'Académie des Sciences - Series I Mathematics*, 304:676–696, 1987.

[31] A.-S. Bonnet-Ben Dhia, C. Hazard, and S. Lohrengel. A singular field method for the solution of Maxwell's Equations in polyhedral domains. *SIAM Journal on Applied Mathematics*, 59:2028–2044, 1999.

[32] M. Dobrowolski. *Numerical Approximation of Elliptic Interface and Corner Problems*. Habilitationschrift, 1981.

[33] L.C. Evans. *Partial Differential Equations*. American Mathematical Society, 2010.

[34] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, 1998.

[35] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*. Springer-Verlag, 1986.

[36] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman Advanced Publishing Program, 1985.

[37] P. Grisvard. *Singularities in Boundary Value Problems*. Masson, 1992.

[38] W. Hackbusch. *Multi-grid Methods and Applications*. Springer-Verlag, 1985.

[39] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, 11:237–339, 2002.

[40] J. Jin. *Theory and Computation Of Electromagnetic Fields*. John Wiley & Sons, Inc., 2010.

[41] R.B. Kellogg. *Singularities in interface problems, in: B. Hubbard (Ed.), Numerical Solution of Partial Differential Equations II*. Academic Press, 1971.

[42] R.B. Kellogg. *Higher order singularities for interface problems, in: A.K. Aziz (Ed.), The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equation*. Academic Press, 1972.

[43] R.B. Kellogg. On the Poisson equation with intersecting interfaces. *Applied Analysis*, 4:101–129, 1975.

[44] V.A. Kondrat'ev. Boundary problems for elliptic equations with conical or angular Points . *Transactions of the Moscow Mathematical Society*, 16:227–313, 1967.

[45] D. Leguillon and E. Sanchez-Palencia. *Computation of Singular Solutions in Elliptic Problems and Elasticity.* Masson, 1987.

[46] P.K. Lemrabet. Régularité de la solution d'un probléme de transmission. *Journal de Mathématiques Pures et Appliquées*, pages 1–38, 1977.

[47] J. Mandel. Multigrid convergence for nonsymmetric, indefinite variational problems and one smoothing step. *Applied Mathematics and Computation*, 19:201–216, 1986.

[48] S. McCormick. *Multigrid Methods.* SIAM, 1987.

[49] P. Monk. *Finite Element Methods for Maxwell's Equations.* Oxford University Press, 2003.

[50] S.A. Nazarov and B.A. Plamenevsky. *Elliptic Problems in Domains with Piecewise Smooth Boundaries.* Expositions in Mathematics, vol. 13, de Gruyter, 1994.

[51] J.C. Nédélec. Mixed finite elements in $\mathbb{R}^3$. *Numerische Mathematik*, 35:315–341, 1980.

[52] J.C. Nédélec. A new family of mixed finite elements in $\mathbb{R}^3$. *Numerische Mathematik*, 50:57–81, 1986.

[53] S. Nicaise. *Polygonal Interface Problems.* Peter Lang, 1993.

[54] S. Nicaise and A.M. Sändig. General interface problems I. *Mathematical Methods in the Applied Sciences*, 17:395–429, 1994.

[55] S. Nicaise and A.M. Sändig. General interface problems II. *Mathematical Methods in the Applied Sciences*, 17:431–450, 1994.

[56] S. Nicaise and A.M. Sändig. Transmission problems for the Laplace and elasticity operators: Regularity and boundary integral formulation. *Mathematical Models and Methods in Applied Science*, 9:855–898, 1999.

[57] W. Rudin. *Principles of Mathematical Analysis.* McGraw-Hill Science/Engineering/Math, 1976.

[58] A. Schatz. An observation concerning Ritz-Galerkin methods with indefinite bilinear forms. *Mathematics of Computation*, 28:958–962, 1974.

[59] L.R. Scott and S. Zhang. Finite element interpolation of non-smooth functions satisfying boundary conditions. *Mathematics Of Computation*, 54:483–493, 1990.

[60] J.A. Stratton. *Electromagnetic Theory*. Swedenborg Press, 2010.

[61] B.A. Szabó and I. Babuska. Computation of the amplitude of stress singular terms for cracks and reentrant corners. *Fracture Mechanics: Nineteenth Symposium*, pages 101–124, 1988.

[62] Ch. Weber and P. Werner. A local compactness theorem for Maxwell's equations. *Mathematical Methods in the Applied Sciences*, 2:12–25, 1980.

[63] J. Xu. A new class of iterative methods for nonselfadjoint or indefinite problems. *SIAM Journal on Numerical Analysis*, 29:303–319, 1992.

[64] J. Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review*, 34:581–613, 1992.

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